Fibonacci arrays, Beatty sequences and Wythoff Nim

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This material is a joint work with Peter Hegarty, math GU. It stemes from the last chapter of Jonas Knapes, and my master thesis (2004), with Peter as supervisor.

Our motivation: Two years ago we encountered for the first time concept of permutations avoiding arithmetic patterns, for example we studied the equation $2x = y + z$:

A permutation of the ntural numbers π avoids arithmetic progressions if whenever $x \neq z$ and $2x = y + z$ then $2\pi(x) \neq \pi(y) + \pi(z)$.

In our masters thesis we were pondering on wether $\lim \frac{\pi_g(x)}{x}$ exists and equals 1? (still open)

We are going to study functions on the natural numbers, defined as follows: Let $x, y \in \mathbf{N}$.If $x \neq y$ then

$$\pi(x) - x \neq \pi(y) - y.$$

A remark on notation:

In each case some specific " greedy algorithm" for constructing an injection on $\mathbf{N}$ with a given multiset is our main object of study. Henceforth will always use the notation π_g when we refer to such a "greedy injection" into the natural numbers.

We can define a generalisation of functions avoiding repetition of differences: Given a multisubset D of $\mathbf{Z}$, we would like π to satisfy

$$D = \{\pi(s) - s \mid s \in \mathbf{N}\}.$$

We refer to D as the difference multiset of the function π. At the end of the presentation we will briefly look at π_g for the following examples:

1) $D = \{\ldots, -1, -1, 0, 0, 1, 1, 2, 2, \ldots\}$
2) $D = \{\ldots, -2, 0, 2, 4, \ldots\}$.

For the purpose of this talk let us denote the following injection "Difference greedy".

Definition 1 *Let* $\pi_g : \mathbf{N} \to \mathbf{N}$ *and let* $x, y \in \mathbf{N}$*. Define* $\pi_g(x)$ *recursively as the least natural number such that*

$$\pi_g(x) \notin \{ \pi(y) \mid 1 \leq y < x \}$$

and such that $\pi(x) - x \neq \pi(y) - y$ *for* $1 \leq y < x$*.*

Notation:

Suppose $f : \mathbf{N} \to \mathbf{N}$. We say that x is an *up position* [of f] if $f(x) - x \geq 0$ and x is a *down position* if $f(x) - x \leq 0$. So x is both an up position and a down position iff $f(x) - x = 0$.

Let $d_f : \mathbf{N} \to \mathbf{Z}$ be defined by $d_f(x) := f(x) - x$.

Definition 2 *Let p be a permutation of the natural numbers. Then p has the* uniqueness property, *$\mathcal{U}$, if p satisfies:*

(i) If $x < y$ are two up positions then

$$p(x) - x < p(y) - y.$$

(ii) If $x < y$ are two down positions then

$$p(x) - x > p(y) - y.$$

(iii) d_p is bijective.

Lemma 1 *Difference greedy* π_g
(i) satisfies $\mathcal{U}$*;*
(ii) is an involution of the natural numbers;
(iii) is the unique permutation of the natural numbers satisfying $\mathcal{U}$*.*

We will now answer these questions:

1. What is the asymptotic "behaviour" of π_g?

2. Can one find an "explicit" representation of π_g ?

3. Are there other interpretations of π_g?

Let us look at 3. first.

At least 5 “different” interpretations of π_g are known.

1. The Wythoff-Nim game. (Wythoff 1907)

2. Beatty sequences. (“Beatty” 1926)

3. Stolarsky interspersions [arrays of Fibonacci sequences]. (Morrison 1982)

4. The golden sieve. (Dispersion sequences.)

5. An integer average greedy permutation.

The Wythoff-Nim game (Wythoff 1907) is also known under the name "Corner the queen". This game is a finite impartial 2-person game, played under the usual rules of play, may be best presented "live" on a chessboard.

Lemma 2 *Winning positions [for the previous player]= "Wythoff pairs"* $=\{(x, \pi_g(x)) \mid x \in \mathbf{N})\}$.

Proof." idea". One can recursively define an involution with the Wythoff pairs, starting with $(1,1),(2,3)$, etc. Lemma 1 applies.

We have the following proposition:

Proposition 1 *If f is a permutation of the natural numbers such that $f(x) - x \neq f(y) - y$ whenever $x \neq y$, then*

$$\limsup_{n\to\infty} \frac{f(n)}{n} - \liminf_{n\to\infty} \frac{f(n)}{n} \geq 1.$$

It is not hard to see that any permutation that satisfies $\mathcal{U}$ is in the sense of this proposition, asymptotically the "tightest of its kind". Hence:

Using our previous notation, putting

$$L = L_{\phi} = \limsup_{n \to \infty} \frac{\pi(n)}{n}$$

and

$$l = l_{\phi} = \liminf_{n \to \infty} \frac{\pi(n)}{n},$$

we get that:

Proposition 2

$$L - l = 1.$$

Since we know that π_g is an involution of $\mathbf{N}$, we also have

$$L = 1/l.$$

Combining these two results give the equation

$$L^2 - L - 1 = 0,$$

which has the positive solution

$$L = \frac{1+\sqrt{5}}{2} =: \Phi,$$

known as the golden ratio.

For a "closed form" of π_g we need a result known as Beattys theorem:

Theorem 1 *Let r and s be positive irrational numbers such that*

$$\frac{1}{r} + \frac{1}{s} = 1.$$

Further, let

$$A = \{\lfloor nr \rfloor \mid n \in \mathbf{N}\}$$

and

$$B = \{\lfloor ns \rfloor \mid n \in \mathbf{N}\}.$$

Then $A \cap B = \emptyset$ and $A \cup B = \mathbf{N}$.

Lemma 3 *As before, let* ϕ *denote the golden ratio and let* $g : \mathbf{N} \to \mathbf{N}$ *be be defined accordingly. Put* $g(1) = 1$ *and for* $n \in \mathbf{N}$ *put* $g(\lceil n\phi \rceil) = \lceil n\phi^2 \rceil$ *and* $g(\lceil n\phi^2 \rceil) = \lceil n\phi \rceil$*. Then* g *satisfies* $\mathcal{U}$*.*

Proof. Theorem 1 implies that g is welldefined as an involution and hence a permutation of the natural numbers.The conditions (i) and (ii) of $\mathcal{U}$ are immediate form the definition. For (iii) notice that $d_g : \mathbf{N} \to \mathbf{Z}$ is bijective, since $\lceil (n+1)\phi^2 \rceil - \lceil (n+1)\phi \rceil = \lceil n\phi^2 + \phi^2 \rceil - \lceil n\phi + \phi \rceil = 1 + \lceil n\phi^2 \rceil - \lceil n\phi \rceil$ and $d_g(1) = 0$. We are done. $\square$

Corollary 1 $g = \pi_g$.

“Hands on work” on Interspersions and the Wythoff array:

Every result below is due to Clark Kimberling in articles published in the 1990:s, except for Morrison’s discovery of the Wythoff pairs in the Wythoff array. We will make intuitive “definitions” on the blackboard of the following concepts, exept 1 (slide).

1. What is an interspersion?

2. Every interspersion is a dispersion [of a strictly increasing sequence].

3. Interspersions are in a natural one-to-one correspondence with fractal sequences.

4. Stolarsky interspersions are in a natural one-to-one correspondence to binary sequences.

They are a subclass of interspersions, defined recursively from Fibonacci sequences.

5. The Wythoff array and the Dual of the Wythoff array are two important examples of Stolarsky arrays. They both consist entirely of Wythoff pairs.

6. The Zeckendorff array equals the Wythoff array.

For 2. we will use "the golden sieve" to arrive at the Wythoff array.

My impression: There are at least one trillion ways to arrive at the Wythoff array. At first we give a formal definition of interspersions:

Definition 3 *An array* $A = (a_{ij})$ *is an* inter-spersion *if*

(i) the rows of A *comprise a partition of the natural numbers;*
(ii) every row of A *is an increasing sequence;*
(iii) every column of A *is an increasing (possibly finite) sequence;*
(iv) if (u_i) *and* (v_j) *are distinct rows of* A *and if* p *and* q *are indices for which* $u_p < v_q < u_{p+1}$, *then* $u_{p+1} < v_{q+1} < u_{p+2}$.

Then (iv) inductively implies, given any two rows of A, beginning at the first term of the row having the greater initial term, all the following terms are individually separated by individual terms of the other row.

An integer-average greedy permutation:

We can define a permutation in a greedy way as taking the least natural number such that the average of the "initial segment" is an integer. More precisely:

Definition 4 *For* $n \in \mathbf{N}$, *let* $\sigma(n)$ *be the least natural number such that* $\sigma(n) \neq \sigma(i)$ *for all* $i < n$ *and such that*

$$(\sum_{i=1}^{n} \sigma(i))/n$$

is an integer.

Proposition 3 $\sigma = \pi_g$.

Proof. The idea is to show that σ satisfies $\mathcal{U}$.

We will now look at possible directions of generalising our "difference greedy algorithm" (little in the literature). In each case, the first two questions for us to answer is if π_g satisfies $\mathcal{U}$ and if $\pi_g = \pi_g^{-1}$?

1. Allow no more then $N \in \mathbf{N}$ repetitions of $\pi_g(x) - x$, otherwise the same as before. The answers are yes and yes.

2. Only allow differences $\pi_g(x) - x$, belonging to a set $S \subset \mathbf{Z}$, say $S =$ *all even integers*. Then if $S = -S$ π_g is an involution of **N**, the answers are yes and yes.

3. In general, as long as $S = -S$ is a multisubset of Z, then π_g is an involution satisfying a certain uniqueness property (depending only on S).

4. Another point of view: Only avoid repetitions of differences in certain positions, for example the even positions (which can be viewed as avoiding repetitions of $\pi(2x) - 2x$). However the direct analog of our "difference greedy algorithm", does not give an involution and hence does not satisfy any uniqueness property. The answers are no and no. But inspite of the greater complexity we have found an explicit construction for $\pi_g := Gr_2$ which we present next;

5. We have a conjecture for an "explicit" construction of "a greedy algorithm that never repeats $\pi_g(kx) - kx$" (denote with Gr_k).

Notation: For any positive integer n we denote

$$\epsilon_n := \sqrt{3}n - \lfloor \sqrt{3}n \rfloor.$$

Set

$$\delta := 2 - \sqrt{3},$$

and observe that, for all n,

$$\epsilon_n - \epsilon_{n+1} \equiv \delta \pmod{1}. \tag{1}$$

Definition : We define a function $f = f_2 : \mathbf{N} \to \mathbf{N}$ as follows :

(I) $f(1) := 1$,

(II) for any $n > 1$,

$f(2n-1) := \min\{t : t \neq f(i) \text{ for any } i \leq 2n-2\}$.

(III) for any $n > 0$,

$f(2n) := n + \lfloor\sqrt{3}n\rfloor$, if $\epsilon_n > \delta$ and $\epsilon_{n-1} > \delta$,

$f(2n) := n + \lfloor\sqrt{3}n\rfloor$,

if $\epsilon_n < \delta$ and $\lfloor\sqrt{3}n\rfloor \in \{f(i) : i < 2n\}$,

$f(2n) := \lfloor\sqrt{3}n\rfloor$,

if $\epsilon_n < \delta$ and $\lfloor\sqrt{3}n\rfloor \notin \{f(i) : i < 2n\}$,

$f(2n) := n + \lfloor\sqrt{3}n\rfloor$,

if $\epsilon_{n-1} < \delta$ and $f(2n-2) = \lfloor\sqrt{3}(n-1)\rfloor$,

$f(2n) := \lfloor\sqrt{3}n\rfloor + 2$,   otherwise, i.e.: iff

$\epsilon_{n-1} < \delta$ and $\lfloor\sqrt{3}(n-1)\rfloor \in \{f(i) : i < 2n-2\}$.

It is clear that the function f is well-defined.

Theorem $f = Gr_2$.

A general result: Let $S \subset \mathbf{Z}$. We define

$$d(S) = \lim_{n\to\infty} \frac{|S \cap [-n, n]|}{2n+1}.$$

We can view S as a set of numbers avoiding a linear equation.

Proposition 4 *Say $d(S) = d$. Then, if*

$$| S \cap -\mathbf{N} |= \infty$$

we have that π_g is a permutation.

If $S = -S$ then $\pi_g = \pi_g^{-1}$.

With L and l as before we get $L = 1/l, L - l = 1/d$ and $L^2 = L/d + 1$.

The remainder of these slides should be seen as an informal sketch of ongoing work.

In a more general setting, what answers are there to the questions:

1. What is the asymptotic "behaviour" of π_g?

2. Can one find an "explicit" representation of π_g ?

3. Are there other interpretations of π_g?

For Gr_2, "yes" to 2. and for Gr_k, heuristics on $k > 1$ gives asymptotics $L = \limsup$ as the root of $L^2 = L + 1/k$ and then $\liminf = l$ from $L - l = 1/k$. Observe in this case, $L \neq 1/l$.

Notation:

Let a_n denote the number of occurances of n in a multiset M. Write:

$M = mZ$ if [$a_n = 1$ if $m|n, a_n = 0$ otherwise] and $M = m * Z$ if [$a_n = m$ for all m].

1. There exist $\lim_{\infty \leftarrow n < f(n)} \pi_g(n)/n =$ Quadratic irrational and $\lim_{\infty \leftarrow n > f(n)} \pi_g(n)/n =$ Quadratic irrational iff $S = -S$ has density > 0. (Otherwise $\limsup = \infty$ and $\liminf = 0$.)

2. Yes for $S = mZ$ (Beatty sequences), Fraenkel (1982). For $N > 1$ and $S = N * Z$, probably, at least "semi-explicit" representations. We have a 4-value conjecture:

$$\pi_g \in \{\lfloor n\phi^N \rfloor, \lfloor n\phi^N \rfloor + 1,$$

$$\lfloor n(\phi^N - 1/N) \rfloor, \lfloor n(\phi^N - 1/N) \rfloor + 1\},$$

where ϕ^N is the positive solution to $x^2 - x/N - 1 = 0$.

3. Definitely yes, see next slide.

This is what we know so far:

1. The Wythoff-Nim game: yes for sZ and $n * Z$.

2. Beatty sequences: Probably for both sZ and $n * Z$.

3. The Wythoff array: yes for sZ and $n * Z$.

4. The golden sieve: don't know.

5. An integer average greedy permutation. yes for $n * Z$, but here not an "entirely greedy" interpretation.