2-PILE NIM WITH A RESTRICTED NUMBER OF MOVE-SIZE IMITATIONS

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Abstract

We study a variation of the combinatorial game of 2-pile Nim. Move as in 2-pile Nim but with the following constraint: Suppose the previous player has just removed say $x > 0$ tokens from the shorter pile (either pile in case they have the same height). If the next player now removes x tokens from the larger pile, then he imitates his opponent. For a predetermined natural number p, by the rules of the game, neither player is allowed to imitate his opponent on more than $p - 1$ consecutive moves. We prove that the strategy of this game resembles closely that of a variant of Wythoff Nim—a variant with a blocking manoeuvre on $p - 1$ diagonal positions. In fact, we show a slightly more general result in which we have relaxed the notion of what an imitation is. The paper includes an appendix by Peter Hegarty, Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, hegarty@chalmers.se.

1. Introduction

A finite impartial game is usually a game where

- there are 2 players and a starting position,

- there is a finite set of possible positions of the game,

- there is no hidden information,

- there is no chance-device affecting how the players move,

- the players move alternately and obey the same game rules,

- there is at least one final position, from which a player cannot move, which determines the winner of the game, and

- the game ends in a finite number of moves, no matter how it is played.

If the winner of the game is the player who makes the final move, then we play under normal play rules; otherwise we play a misère version of the game.

In this paper *a game*, say G, is always a finite impartial game played under normal rules. The player who made the most recent move will be denoted by *the previous*

player. A position from which the previous player will win, given best play, is called a *P-position*, or just P. A position from which the *next player* will win is called an *N-position*, or just N. The set of all P-positions will be denoted by $\mathcal{P} = \mathcal{P}(G)$ and the set of all N-positions by $\mathcal{N} = \mathcal{N}(G)$.

Suppose A and B are the two piles of a 2-pile take-away game, which contain $a \geq 0$ and $b \geq 0$ tokens respectively. Then the *position* is (a, b) and a *move* (or an *option*) is denoted by $(a, b) \rightarrow (c, d)$, where $a - c \geq 0$ and $b - d \geq 0$ but not both $a = c$ and $b = d$. All our games are symmetric in the sense that (a, b) is P if and only if (b, a) is P. Hence, to simplify notation, when we say (a, b) is P (N) we also mean (b, a) is P (N). Throughout this paper, we let $\mathbb{N}_0$ denote the non-negative integers and $\mathbb{N}$ the positive integers.

1.1. The Game of Nim

The classical game of Nim is played on a finite number of piles, each containing a non-negative finite number of tokens, where the players alternately remove tokens from precisely one of the non-empty piles—that is, at least one token and at most the entire pile—until all piles are gone. The winning strategy of Nim is, whenever possible, to move so that the "Nim-sum" of the pile-heights equals zero, see for example [4] or [22, page 3]. When played on one single pile there are only next player winning positions except when the pile is empty. When played on two piles, the pile-heights should be equal to ensure victory for the previous player.

1.2. Adjoin the P-positions as Moves

A possible extension of a game is $(\star)$ *to adjoin the P-positions of the original game as moves in the new game*. Clearly this will alter the P-positions of the original game.

Indeed, if we adjoin the P-positions of 2-pile Nim as moves, then we get another famous game, namely Wythoff Nim (a.k.a. Corner the Queen), see [25]. The set of moves are: Remove any number of tokens from one of the piles, or remove the same number of tokens from both piles.

The P-positions of this game are well-known. Let $\phi = \frac{1+\sqrt{5}}{2}$ denote the golden ratio. Then (x, y) is a P-position if and only if

$$(x, y) \in \left\{ \left(\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor \right) \mid n \in \mathbb{N}_0 \right\}.$$

We will, in a generalized form, return to the nice arithmetic properties of this and other sequences in Proposition 14 (see also [13] for further generalizations).

Other examples of $(\star)$ are the Wythoff-extensions of n-pile Nim for $n \geq 3$ discussed in [3, 8, 23, 24] as well as some extensions to the game of 2-pile Wythoff Nim in [9], where the authors adjoin subsets of the Wythoff Nim P-positions as moves in new games.

1.3. Remove a Game's Winning Strategy

There are other ways to construct interesting extensions to Nim on just one or two piles. For example we may introduce a so-called *move-size dynamic* restriction, where the options in some specific way depend on how the previous player moved (for example how many tokens he removed), or "pile-size dynamic"[1] restrictions, where the options depend on the number of tokens in the respective piles.

The game of "Fibonacci Nim" in [2, page 483] is a beautiful example of a move-size dynamic game on just one pile. This game has been generalized, for example in [16]. Treatments of two-pile move-size dynamic games can be found in [5], extending the (pile-size dynamic) "Euclid game," and in [14].

The games studied in this paper are move-size dynamic. In fact, similar to the idea in Section 1.2, there is an obvious way to alter the P-positions of a game, namely ($\star\star$) *from the original game, remove the next-player winning strategy.* For 2-pile Nim this means that we remove the possibility to *imitate* the previous player's move, where imitate has the following interpretation:

Definition 1 Given two piles, A and B, where $\#A \leq \#B$ and the number of tokens in the respective pile is counted before the previous player's move, then, if the previous player removed tokens from pile A, the next player *imitates* the previous player's move if he removes the same number of tokens from pile B as the previous player removed from pile A.

We call this game *Imitation Nim.* The intuition is, given the position (a, b), where $a \leq b$, Alice can prevent Bob from going to (c, d), where $c < a$ and $b - a = d - c$, by moving $(a, b) \rightarrow (c, b)$. We illustrate with an example:

Example 2 Suppose the position is $(1, 3)$. If this is an initial position, then there is no 'dynamic' restriction on the next move so that the set $\{(1, 2), (1, 1), (1, 0), (0, 3)\}$ of Nim options is identical to the set of Imitation Nim options. But this holds also, if the previous player's move was

$$(1, x) \rightarrow (1, 3),$$

or

$$(x, 3) \rightarrow (1, 3) \tag{1}$$

where $x \geq 4$. For these cases, the imitation rule does not apply since the previous player removed tokens from the larger of the two piles. If, on the other hand, the previous move was as in (1) with $x \in \{2, 3\}$ then, by the imitation rule, the option $(1, 3) \rightarrow (1, 3 - x + 1)$ is prohibited.

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[1] We understand that pile-size dynamic games are not 'truly' dynamic since for any given position of a game, one may determine the P-positions without any knowledge of how the game has been played up to that point.

Further, $(3,3) \to (1,3)$ is a losing move, since, as we will see in Proposition 5 (i), $(1,3) \to (1,2)$ is a winning move. But, by the imitation rule, $(2,3) \to (1,3)$ is a winning move, since for this case $(1,3) \to (1,2)$ is forbidden.

This last observation leads us to ask a general question for a move-size dynamic game, roughly: *When does the move-size dynamic rule change the outcome of a game?* To clarify this question, let us introduce some non-standard terminology, valid for any move-size dynamic game.

Definition 3 Let G be a move-size dynamic game. A position $(x,y) \in G$ is

(i) *dynamic*: if, in the course of the game, we cannot tell whether it is P or N without knowing the history, at least the most recent move, of the game,

(ii) *non-dynamic*

   P: if it is P regardless of any previous move(s),
   N: ditto, but N.

Remark 4 Henceforth, if not stated otherwise, we will think of a (move-size dynamic) game as a game where the progress towards the current position is memorized in an appropriate manner. A consequence of this approach is that each (dynamic) position is P or N.

In light of these definitions, we will now characterize the winning positions of a game of Imitation Nim (see also Figure 1). This is a special case of our main theorem in Section 2. Notice, for example, the absence of Wythoff Nim P-positions that are dynamic, considered as positions of Imitation Nim.

Proposition 5 *Let $0 \le a \le b$ be integers. Suppose the game is Imitation Nim. Then (a,b) is*

(i) *non-dynamic P if and only if it is a P-position of Wythoff Nim;*

(ii) *non-dynamic N if and only if*

   (a) *there are integers $0 \le c \le d < b$ with $b - a = d - c$ such that (c,d) is a P-position of Wythoff Nim, or*

   (b) *there is a $0 \le c < a$ such that (a,c) is a P-position of Wythoff Nim.*

Remark 6 Given the notation in Proposition 1, it is well-known (see also Figure 1) that if there is an $x < a$ such that (x,b) is a P-position of Wythoff Nim, then this implies the statement in (iia). One may also note that, by symmetry, there is an intersection of type (iia) and (iib) positions, namely whenever $a = d$, that is whenever $c < a < b$ is an arithmetic progression.
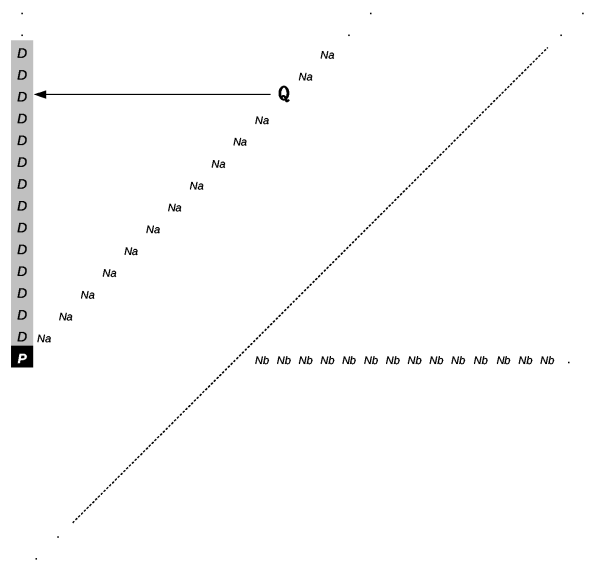
Figure 1: The strategy of Imitation Nim. The P is a (Wythoff Nim) P-position north of the main diagonal. The D's are dynamic positions. The arrow symbolizes a winning move from **Q**. The Na's are the positions of type (iia) in Proposition 1, the Nb's of type (iib).

By Proposition 5 and Remark 6, (c, b) is a dynamic position of Imitation Nim if and only if there is a P-position of Wythoff Nim, (c, d), with $c \leq d < b$. Further, with notation as in (iia), we get that (c, b) is dynamic P if and only if the previous player moved $(a, b) \rightarrow (c, b)$, for some $a > c$.

Recall that the first few P-positions of Wythoff Nim are

$$(0, 0), (1, 2), (3, 5), \dots.$$

Hence, in Example 1, a (non-dynamic) P-position of Imitation Nim is $(1, 2)$. The position $(1, 3)$ is (by Example 1 or by the comment after Remark 2) dynamic. The positions $(2, 3)$ and $(3, 4)$ are, by Proposition 1 (iia), non-dynamic N. As examples of non-dynamic N-positions of type (iib) we may take $(2, x)$ with $x \geq 3$.

By the comment after Remark 2, we get:

Corollary 7 *Treated as initial positions, the P-positions of Imitation Nim are identical to those of Wythoff Nim.*

Remark 8 For a given position, the rules of Wythoff Nim allow more options than those of Nim, whereas the rules of Imitation Nim give fewer. Nevertheless, the P-

positions are identical if one only considers starting positions. Hence, one might want to view these variants of 2-pile Nim as each other's "duals."

1.4. Two Extensions of Imitation Nim and Their "Duals"

We have given a few references for the subject of move-size dynamic games, of which the first is [2]. But literature on our next topic, *games with memory*, seems to appear only in a somewhat different context[2] from that which we shall develop.

1.4.1. A Game with Memory

A natural extension of Imitation Nim is, given $p \in \mathbb{N}$, to allow $p - 1$ consecutive imitations by one and the same player, but to prohibit the pth imitation. We denote this game by $(1, p)$-*Imitation Nim*.

Remark 9 This rule removes the winning strategy from 2-pile Nim if and only if the number of tokens in each pile is $\geq p$.

Example 10 Suppose the game is $(1, 2)$-Imitation Nim, so that no two consecutive imitations by one and the same player are allowed. Suppose the starting position is $(2, 2)$ and that Alice moves to $(1, 2)$. Then, if Bob moves to $(1, 1)$, Alice will move to $(0, 1)$, which is P for a game with this particular history. This is because the move $(0, 1) \to (0, 0)$ would have been a second consecutive imitation for Bob and hence is not permitted. If Bob chooses instead to move to $(0, 2)$, then Alice can win in the next move, since $2 > 1$ and hence the imitation rule does not apply.

Indeed, Alice's first move is a winning move, so $(2, 2)$ is N (which is non-dynamic) and $(1, 2)$ is P. But if $(1, 2)$ had been an initial position, then it would have been N, since $(1, 2) \to (1, 1)$ would have been a winning move. So $(1, 2)$ is dynamic. Clearly $(0, 0)$ is non-dynamic P. Otherwise the 'least' non-dynamic P-position is $(2, 3)$, since $(2, 2)$ is N and $(2, 1)$ or $(1, 3) \to (1, 1)$ would be winning moves, as would $(2, 0)$ or $(0, 3) \to (0, 0)$.

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[2]The following discussion on this subject is provided by our anonymous referee:

Kalmár [17] and Smith [21] defined a *strategy in the wide sense* to be a strategy which depends on the present position and on all its antecedents, from the beginning of play. Having defined this notion, both authors concluded that it seems logical that it suffices to consider a *strategy in the narrow sense*, which is a strategy that depends only on the present position (analogous to a *Markov chain*, where only the present position determines the next). They then promptly restricted attention to strategies in the narrow sense.

Let us define a *strategy in the broad sense* to be a strategy that depends on the present position v and on *all* its predecessors $u \in F^{-1}(v)$, whether or not such u is a position in the play of the game. This notion, if anything, seems to be even less needed than a strategy in the wide sense.

Yet, in [10], a strategy in the broad sense was employed for computing a winning move in polynomial time for annihilation games. It was needed, since the counter function associated with γ (=generalized Sprague-Grundy function) was computed only for a small subgraph of size $O(n^4)$ of the game-graph of size $O(2^n)$, in order to preserve polynomiality. This suggests the possibility that a polynomial strategy in the narrow sense may not exist; but this was not proved. It is only reported there that no polynomial time strategy in the narrow sense was found.

1.4.2. The Dual of $(1, p)$-Imitation Nim

In [13, 18] we put a *Muller twist* or *blocking manoeuvre* on the game of Wythoff Nim. A nice introduction to games with a Muller twist (comply/constrain games) is given in [22]. Variations on Nim with a Muller twist can also be found, for example, in [11] (which generalizes a result in [22]), [15] and [26].

Fix $p \in \mathbb{N}$. The rules of the game which we shall call $(1, p)$-*Wythoff Nim* are as follows. Suppose the current pile position is (a, b). Before the next player removes any tokens, the previous player is allowed to announce $j \in \{1, 2, \ldots, p-1\}$ positions, say $(a_1, b_1), \ldots, (a_j, b_j)$ where $b_i - a_i = b - a$, to which the next player may not move. Once the next player has moved, any blocking manoeuvre is forgotten. Otherwise move as in Wythoff Nim.

We will show that as a generalization of Corollary 1, if X is a starting position of $(1, p)$-Imitation Nim then it is P if and only if it is a P-position of $(1, p)$-Wythoff Nim. A generalization of Proposition 1 also holds, but let us now move on to our next extension of Imitation Nim.
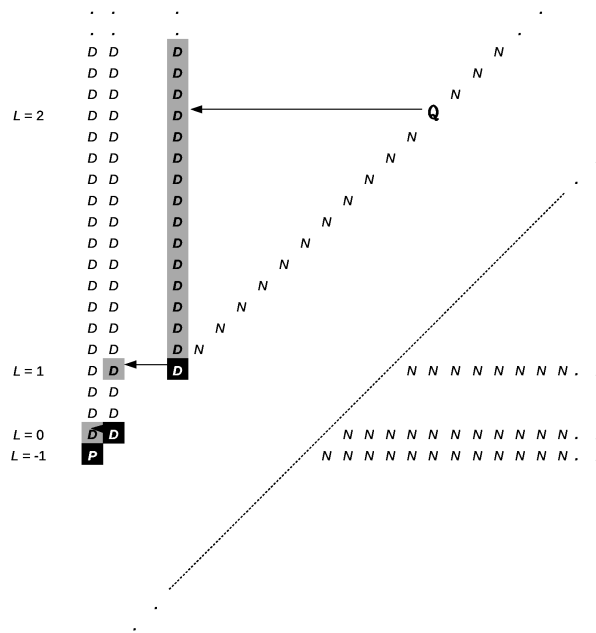

Figure 2: The strategy of $(1, 3)$-Imitation Nim. The P is a non-dynamic P-position north of the main diagonal. The black positions are all P-positions of $(1, 3)$-Wythoff Nim on one and the same SW-NE diagonal. The D's are dynamic positions. The arrows symbolize three consecutive winning moves from a position **Q**. The N's are non-dynamic N-positions.

1.4.3. A Relaxed Imitation

Let $m \in \mathbb{N}$. We relax the notion of an imitation to an *m-imitation* (or just imitation) by saying: provided the previous player removed x tokens from pile A, with notation as in Definition 1, then the next player m-imitates the previous player's move if he removes $y \in \{x, x+1, \ldots, x+m-1\}$ tokens from pile B.

Definition 11 Fix $m, p \in \mathbb{N}$. We denote by (m, p)-*Imitation Nim* the game where no p consecutive m-imitations are allowed by one and the same player.

Example 12 Suppose that the game is $(2, 1)$-Imitation Nim, so that no 2-imitation is allowed. Then if the starting position is $(1, 2)$ and Alice moves to $(0, 2)$, Bob cannot move, hence $(1, 2)$ is an N-position and it must be non-dynamic since $(1, 2) \to (0, 2)$ is always an option regardless of whether there was a previous move or not.

1.4.4. The Dual of $(\mathbf{m}, \mathbf{1})$-Imitation Nim

Fix a positive integer m. There is a generalization of Wythoff Nim, see [6], here denoted by $(m, 1)$-Wythoff Nim, which (as we will show in Section 2) has a natural P-position correspondence with $(m, 1)$-Imitation Nim. The rules for this game are: remove any number of tokens from precisely one of the piles, or remove tokens from both piles, say x and y tokens respectively, with the restriction that $|x - y| < m$.

And indeed, to continue Example 3, $(1, 2)$ is certainly an N-position of $(2, 1)$-Wythoff Nim, since here $(1, 2) \to (0, 0)$ is an option. On the other hand $(1, 3)$ is P, and non-dynamic P of $(1, 2)$-Imitation Nim. For, in the latter game, if Alice moves $(1, 3) \to (0, 3)$ or $(1, 0)$, it does not prevent Bob from winning and $(1, 3) \to (1, 2)$ or $(1, 1)$ are losing moves, since Bob may take advantage of the imitation rule.

In [6], the author shows that the P-positions of $(m, 1)$-Wythoff Nim are so-called "Beatty pairs" (view for example the appendix, the original papers in [20, 1] or [6, page 355], of the form $(\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor)$, where $\beta = \alpha + m$, n is a non-negative integer and

$$\alpha = \frac{2 - m + \sqrt{m^2 + 4}}{2}. \tag{2}$$

1.4.5. The P-positions of $(\mathbf{m}, \mathbf{p})$-Wythoff Nim

In the game of (m, p)-Wythoff Nim, originally defined in [13] as p-blocking m-Wythoff Nim, a player may move as in $(m, 1)$-Wythoff Nim and block positions as in $(1, p)$-Wythoff Nim. From this point onwards, whenever we write Wythoff's game or $W = W_{m,p}$, we are referring to (m, p)-Wythoff Nim.

The P-positions of this game can easily be calculated by a minimal exclusive algorithm (but with exponential complexity in succinct input size) as follows: Let X be a set of non-negative integers. Define $\text{mex}(X)$ as the least non-negative integer not in X, formally $\text{mex}(X) := \min\{x \mid x \in \mathbb{N}_0 \setminus X\}$.

Definition 13 Given positive integers m and p, the integer sequences (a_n) and (b_n) are

$$
\begin{aligned}
a_n &= \operatorname{mex}\{a_i, b_i \mid 0 \le i < n\}, \\
b_n &= a_n + \delta(n),
\end{aligned}
$$

where $\delta(n) = \delta_{m,p}(n) := \left\lfloor \frac{n}{p} \right\rfloor m$.

The next result follows almost immediately from this definition. See also [13, Proposition 3.1 and Remark 1] for further extensions.

Proposition 14 ([13]) *Let $m, p \in \mathbb{N}$.*

(a) *The P-positions of (m,p)-Wythoff Nim are the pairs (a_i, b_i) and (b_i, a_i), $i \in \mathbb{N}_0$, as in Definition 13;*

(b) *The sequences $(a_i)_{i \ge 0}^{\infty}$ and $(b_i)_{i \ge p}^{\infty}$ partition $\mathbb{N}_0$ and for $i \in \{0, 1, \ldots, p-1\}$, $a_i = b_i = i$;*

(c) *Suppose (a, b) and (c, d) are two distinct P-positions of (m,p)-Wythoff Nim with $a \le b$ and $c \le d$. Then $a < c$ implies $b - a \le d - c$ (and $b < d$);*

(d) *For each $\delta \in \mathbb{N}$, if $m \mid \delta$ then $\#\{i \in \mathbb{N}_0 \mid b_i - a_i = \delta\} = p$, otherwise $\#\{i \in \mathbb{N}_0 \mid b_i - a_i = \delta\} = 0$.*

The (m,p)-Wythoff pairs from Proposition 14 may be expressed via Beatty pairs if and only if $p \mid m$. In that case one can prove via an inductive argument that the P-positions of (m,p)-Wythoff Nim are of the form

$$
(pa_n,\, pb_n),\ (pa_n + 1,\, pb_n + 1),\ \ldots,\ (pa_n + p - 1,\, pb_n + p - 1),
$$

where (a_n, b_n) are the P-positions of the game $(\frac{m}{p}, 1)$-Wythoff Nim (we believe that this fact has not been recognized elsewhere, at least not in [13] or [12]).

For any other m and p we did not have a polynomial time algorithm for telling whether a given position is N or P, until recently. While reviewing this article there has been progress on this matter, so there is a polynomial time algorithm, see [12]. See also a conjecture in [13], Section 5, saying in a specific sense that the (m,p)-Wythoff pairs are "close to" the Beatty pairs $(\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor)$, where $\beta = \alpha + \frac{m}{p}$ and

$$
\alpha = \frac{2p - m + \sqrt{m^2 + 4p^2}}{2p},
$$

which is settled for the case $m = 1$ in the appendix. In the general case, as is shown in [12], the explicit bounds for a_n and b_n are

$$
(n - p + 1)\alpha \le a_n \le n\alpha
$$

and

$$
(n - p + 1)\beta \le b_n \le n\beta.
$$

A reader who, at this point, feels ready to plough into the main idea of our result, may move on directly to Section 2. There we state how the winning positions of (m, p)-Imitation Nim coincide with those of (m, p)-Wythoff Nim and give a complete proof for the case $m = 1$. In Section 3 we finish off with a couple of suggestions for future work.

1.4.6. Further Examples

In this section we give two examples of games where $p > 1$ and $m > 1$ simultaneously, namely $(2, 3)$- and $(3, 3)$-Imitation Nim respectively. The style is informal.

In Example 4 the winning strategy (via the imitation rule) is directly analogous to the case $m = 1$. In Example 5 we indicate how our relaxation of the imitation rule changes how a player may take advantage of it in a way that is impossible for the $m = 1$ case. We illustrate why this does not affect the nice coincidence between the winning positions of Imitation Nim and Wythoff's game. Hence these examples may be profitably studied in connection with (a second reading of) the proof of Theorem 1.

Example 15 The first few P-positions of $(2, 3)$-Wythoff Nim are

$$(0, 0), (1, 1), (2, 2), (3, 5), (4, 6), (7, 9), (8, 12), (10, 14), (11, 15), (13, 19).$$

For the moment assume that the first few non-dynamic P-positions of $(2, 3)$-Imitation Nim are $(0, 0), (3, 5), (8, 12), (13, 19)$. Clearly, a player should at any point aim at moving to such a position. If this is not possible, one could try and move to a P-position of Wythoff's game. But this is not necessarily a good strategy, in particular if by doing so one imitates the other player's move.

Suppose Alice's first move is $(13, 18) \rightarrow (11, 18)$. Then Bob can move to a P-position of Wythoff's game, namely $(11, 18) \rightarrow (11, 15)$. But this is a 2-imitation. Then Alice may move $(11, 15) \rightarrow (10, 15)$ and once again, provided Bob wants to move to a P-position of Wythoff's game, his only choice is $(10, 15) \rightarrow (10, 14)$, which again is a 2-imitation. At this point he has used up the number of permitted 2-imitations and hence Alice may move $(10, 14) \rightarrow (8, 14)$ and she is assured that Bob will not reach the non-dynamic P-position $(8, 12)$.

So, returning to his first move, he investigates the possibility of removing tokens from the pile with 11 tokens. But, however he does this, Alice will be able to reach a P-position of Wythoff's game *without* imitating Bob. Namely, suppose Bob moves to $(x, 18)$ with $x \leq 10$. Then Alice's next move would be to $(x, y) \in \mathcal{P}(W)$, for some $y \leq x + 4$. Clearly this move is not a 2-imitation since $18 - y - (11 - x) = 7 - (y - x) \geq 3$.

By this example we see that the imitation rule is an eminent tool for Alice, whereas Bob is the player who 'suffers its consequences'. In the next example Bob tries to get

around his predicament by hoping that Alice would 'rely too strongly' on the imitation rule.

Example 16 The first few P-positions of $(3,3)$-Wythoff Nim are

$$(0,0), (1,1), (2,2), (3,6), (4,7), (5,8), (9,15).$$

Suppose, in a game of $(3,3)$-Imitation Nim, the players have moved

$$\text{Alice} : (6,9) \to (5,9)$$
$$\text{Bob} : (5,9) \to (5,6) \text{ an imitation}$$
$$\text{Alice} : (5,6) \to (4,6)$$
$$\text{Bob} : (4,6) \to (3,6) \text{ no imitation.}$$

Bob will win, in spite of Alice trying to use the imitation rule to her advantage. The mistake is Alice's second move, where she should change her 'original plan' and not continue to rely on the imitation rule.

For the next variation Bob tries to 'confuse' Alice's strategy by 'swapping piles',

$$\text{Alice} : (3,3) \to (2,3)$$
$$\text{Bob} : (2,3) \to (2,1).$$

Bob has imitated Alice's move once. If Alice continues her previous strategy by removing tokens from the smaller pile, say by moving $(2,1) \to (2,0)$, Bob will imitate Alice's move a second time and win. Now Alice's correct strategy is rather to remove token(s) from the larger pile,

$$\text{Alice} : (2,1) \to (1,1)$$
$$\text{Bob} : (1,1) \to (0,1)$$
$$\text{Alice} : (0,1) \to (0,0).$$

Here Alice has become the player who imitates, but nevertheless wins.

2. The Winning Strategy of Imitation Nim

For the statement of our main theorem we use some more terminology.

Definition 17 Suppose the constants m and p are given as in Imitation Nim or in Wythoff's game. Then, if $a, b \in \mathbb{N}_0$,

$$\xi(a,b) = \xi_{m,p}\big((a,b)\big) := \#\big\{ (i,j) \in \mathcal{P}(W_{m,p}) \mid j - i = b - a, \ i < a \big\}.$$

Then according to Proposition 14 (d), $0 \le \xi(a,b) \le p$, and indeed, if $(a,b) \in \mathcal{P}(W_{m,p})$ then $\xi(a,b)$ equals the number of P-positions the previous player has to block off (given that we are playing Wythoff's game) in order to win. In particular, $\xi(a,b) < p$ for $(a,b) \in \mathcal{P}(W_{m,p})$.

Definition 18 Let (a,b) be a position of a game of (m,p)-Imitation Nim. Put

$$L(a,b) = L_{m,p}((a,b)) := p - 1$$

if

(A) (a,b) is the starting position, or

(B) $(c,d) \to (a,b)$ was the most recent move and (c,d) was the starting position, or

(C) the previous move was $(e,f) \to (c,d)$ but the move (or option) $(c,d) \to (a,b)$ is not an m-imitation.

Otherwise, with notation as in (C), put $L(a,b) = L(e,f) - 1$.

Notice that by the definition of (m,p)-Imitation Nim,

$$-1 \le L(a,b) < p.$$

It will be convenient to allow $L(a,b) = -1$, although a player cannot move $(c,d) \to (a,b)$ if it is an imitation and $L(e,f) = 0$. Indeed $L(e,f)$ represents the number of imitations the player moving from (c,d) still has 'in credit'.

Theorem 19. *Let $0 \le a \le b$ be integers and suppose the game is (m,p)-Imitation Nim. Then (a,b) is P if and only if*

(I) $(a,b) \in \mathcal{P}(W_{m,p})$ *and* $0 \le \xi(a,b) \le L(a,b)$*, or*

(II) *there is a* $a \le c < b$ *such that* $(a,c) \in \mathcal{P}(W_{m,p})$ *but* $-1 \le L(a,c) < \xi(a,c) \le p - 1$.

Corollary 20 *If (a,b) is a starting position of (m,p)-Imitation Nim, then it is P if and only if it is a P-position of (m,p)-Wythoff Nim.*

Proof. Put $L(\cdot) = p - 1$ in Theorem 19.   □

By Theorem 19 (I) and the remark after Definition 18 we get that (a,b) is non-dynamic P if and only if $(a,b) \in \mathcal{P}(W)$ and $\xi(a,b) = 0$. On the other hand, if $(a,b) \in \mathcal{N}(W)$ it is dynamic if and only if there is a $a \le c < b$ such that $(a,c) \in \mathcal{P}(W)$. (See also Figure 2.)

Proof of Theorem 19. We only give the proof for the case $m = 1$. In this way we may put a stronger emphasis on the idea of the game, at the expense of technical details. We will make repeated use of Proposition 14 (a) without any further comment.

Suppose (a, b) is as in (I). Then we need to show that, if (x, y) is an option of (a, b) then (x, y) is neither of form (I) nor (II).

But Proposition 14 (b) gives immediately that $(x, y) \in \mathcal{N}(W)$ so suppose (x, y) is of form (II). Then there is a $x \le c < y$ such that $(x, c) \in \mathcal{P}(W)$ and $L(x, c) < \xi(x, c)$. Since, by (I), $\xi(a, b) \le L(a, b)$ and $L(a, b) - 1 \le L(x, c)$ $(\le L(a, b))$ we get that

$$\xi(a, b) \le L(a, b) \le L(x, c) + 1 \le \xi(x, c),$$

which, in case $c - x = b - a$, is possible if and only if $\xi(a, b) = \xi(x, c)$. But then, since, by our assumptions, $(x, c) \in \mathcal{P}(W)$ and $(a, b) \in \mathcal{P}(W)$, we get $(a, b) = (x, c)$, which is impossible.

So suppose that $c - x \ne b - a$. Then, by Proposition 14 (c), $c - x < b - a$. We have two possibilities:

$y = b$: Then if $(x, b) \to (x, c)$ is an imitation of $(a, b) \to (x, b)$, we get $b - c > a - x = b - c$, a contradiction.

$x = a$: For this case the move $(a, y) \to (a, c)$ cannot be an imitation of $(a, b) \to (a, y)$, since the previous player removed tokens from the larger pile. Then $L(a, c) = p - 1 \ge \xi(a, c)$ since, by (II), $(a, c) \in \mathcal{P}(W)$.

Hence we may conclude that if (a, b) is of form (I) then an option of (a, b) is neither of form (I) nor (II).

Suppose now that (a, b) is of form (II). Then $(a, c) \in \mathcal{P}(W)$ is an option of (a, b). But we have $L(a, c) < \xi(a, c)$, so (a, c) is not of form (I). Since $(a, c) \in \mathcal{P}(W)$, by Proposition 14 (b), it cannot be of form (II). But then, since $b > c$, by Proposition 14 (b) and (c), any other option of (a, b), say (x, y), must be an N-position of Wythoff's game. So suppose (x, y) is of form (II). We get two cases:

$y = b$: Then $0 \le x < a$ and there is an option $(x, d) \in \mathcal{P}(W)$ of (x, b) with $x \le d < b$. But by Proposition 14 (b) and (c), we have $d - x \le c - a < b - a$ and hence $(x, b) \to (x, d)$ does not imitate $(a, b) \to (x, b)$. Therefore $L(x, d) = p - 1 \ge \xi(x, d)$, which contradicts the assumptions in (II).

$x = a$: Then $0 \le y < b$. If $y > c$, then $(a, c) \in \mathcal{P}(W)$ is an option of (a, y) and two consecutive moves from the larger pile would give $L(a, c) = p - 1 \ge \xi(a, c)$. Otherwise, by Proposition 14 (b), there is no option of (a, y) in $\mathcal{P}(W)$. In either case one has a contradiction to the assumptions in (II).

We are done with the first part of the proof.

Therefore, for the remainder of the proof, assume that (α, β), $0 \le \alpha \le \beta$, is neither of form (I) nor (II). Then,

(i) if $(\alpha, \beta) \in \mathcal{P}(W)$, this implies $0 \le L(\alpha, \beta) < \xi(\alpha, \beta) \le p - 1$, and

(ii) if there is a $\alpha \leq c < \beta$ such that $(\alpha, c) \in \mathcal{P}(W)$, this implies $0 \leq \xi(\alpha, c) \leq L(\alpha, c) \leq p - 1$.

We need to find an option of (α, β), say (x, y), of form (I) or (II).

If $(\alpha, \beta) \in \mathcal{P}(W)$, then (ii) is trivially satisfied by Proposition 14 (b). Also, $\xi(\alpha, \beta) > 0$ by (i). Hence, there is a position $(x, z) \in \mathcal{P}(W)$ such that $z - x = \beta - \alpha$ with $x \leq z < \beta(= y)$. Then, since $L(\alpha, \beta) < \xi(\alpha, \beta)$, the option (x, β) satisfies (II) (and hence, by the imitation rule, $(\alpha, \beta) \to (x, \beta)$ is the desired winning move).

For the case $(\alpha, \beta) \in \mathcal{N}(W)$ (here (i) is trivially true), suppose $(\alpha, c) \in \mathcal{P}(W)$ with $\alpha \leq c < \beta$. Then (ii) gives $L(\alpha, c) \geq \xi(\alpha, c)$, which clearly holds for example if the most recent move wasn't an imitation. In any case this immediately implies (I).

If $c < \alpha$, with $(\alpha, c) \in \mathcal{P}(W)$, then (ii) holds trivially by Proposition 2 (b). Thus (I) holds, because $(\alpha, \beta) \to (\alpha, c)$ isn't an imitation : if it were, then the previous would necessarily have been from the larger pile.

If $c < \alpha$ with $(c, \beta) \in \mathcal{P}(W)$, then the move $(\alpha, \beta) \to (c, \beta)$ isn't an imitation since tokens have been removed from the smaller pile. Hence $p - 1 = L(c, \beta) \geq \xi(c, \beta)$.

By Proposition 14, the only remaining case for (α, β) an N-position of Wythoff's game is whenever there is a position $(x, z) \in \mathcal{P}(W)$ such that $x < \alpha$ and

$$\beta - \alpha = z - x. \tag{3}$$

We may assume there is no $c < \beta$ such that $(\alpha, c) \in \mathcal{P}(W)$, since we are already done with this case. Then (ii) holds trivially and, by Proposition 14 (b), there must be a $c > \beta$ such that $(\alpha, c) \in \mathcal{P}(W)$. But then, by Proposition 14 (c) and (d), we get $\xi(\alpha, \beta) = p > 0$ and so, since for this case we may take (x, z) such that $p - 1 = \xi(x, z)$, we get $L(x, z) \leq p - 2 < \xi(x, z)$. Then, by (3), $(x, \beta) = (x, y)$ is clearly the desired position of form (II). $\square$

3. Final Questions

Let us finish off with some questions.

- Consider a slightly different setting of an impartial game, namely where the second player does not have perfect information, but the first player (who has) is not aware of this fact. Similar settings have been discussed, for example, in [2, 19]. We may ask, for which games (starting with those we have discussed) is there a simple *second player's strategy* which lets him *learn* the winning strategy of the game *while playing*? By this we mean that if he starts a new 'partie' of the same game at least one move after the first one, he wins.

- Is there a generalization of Wythoff Nim to $n > 2$ piles of tokens (see for example [3, 8, 23, 24]), together with a generalization of 2-pile Imitation Nim, such that the P-positions coincide (as starting positions)?

- Are there other impartial (or partizan) games where an imitation rule corresponds in a natural way to a blocking manoeuvre?
- Can one formulate a general rule as to when such correspondences can be found and when not?

Appendix

Peter Hegarty

The purpose of this appendix is to provide a proof of Conjecture 5.1 of [13] in the case $m = 1$, which is the most natural case to consider. Notation concerning 'multisets' and 'greedy permutations' is consistent with Section 2 of [13]. We begin by recalling

Definition. Let r, s be positive irrational numbers with $r < s$. Then (r, s) is said to be a *Beatty pair* if

$$\frac{1}{r} + \frac{1}{s} = 1. \tag{4}$$

Theorem. Let (r, s) be a Beatty pair. Then the map $\tau : \mathbb{N} \to \mathbb{N}$ given by

$$\tau(\lfloor nr \rfloor) = \lfloor ns \rfloor, \ \ \forall\, n \in \mathbb{N}, \qquad \tau = \tau^{-1},$$

is a well-defined involution of $\mathbb{N}$. If M is the multiset of differences $\pm\{\lfloor ns \rfloor - \lfloor nr \rfloor :$ $n \in \mathbb{N}\}$, then $\tau = \pi_g^M$. M has asymptotic density equal to $(s - r)^{-1}$.

Proof. That τ is a well-defined permutation of $\mathbb{N}$ is Beatty's theorem. The second and third assertions are then obvious. $\qquad\square$

Proposition. *Let $r < s$ be positive real numbers satisfying (4), and let $d :=$ $(s - r)^{-1}$. Then the following are equivalent:*

(i) *r is rational,*

(ii) *s is rational,*

(iii) *d is rational of the form $\frac{mn}{m^2 - n^2}$ for some positive rational m, n with $m > n$.*

Proof. Straightforward algebra exercise. $\qquad\square$

Notation. Let (r, s) be a Beatty pair, $d := (s - r)^{-1}$. We denote by M_d the multisubset of $\mathbb{N}$ consisting of all differences $\lfloor ns \rfloor - \lfloor nr \rfloor$, for $n \in \mathbb{N}$. We denote $\tau_d := \pi_g^{\pm M_d}$.

As usual, for any positive integers m and p, we denote by $\mathcal{M}_{m,p}$ the multisubset of $\mathbb{Z}$ consisting of p copies of each multiple of m and $\pi_{m,p} := \pi_g^{\mathcal{M}_{m,p}}$. We now

denote by $M_{m,p}$ the submultiset consisting of all the positive integers in $\mathcal{M}_{m,p}$ and $\overline{\pi}_{m,p} := \pi_g^{\pm M_{m,p}}$. Thus

$$\overline{\pi}_{m,p}(n) + p = \pi_{m,p}(n + p) \quad \text{for all } n \in \mathbb{N}. \tag{5}$$

Since $\mathcal{M}_{m,p}$ has density p/m, there is obviously a close relation between $M_{m,p}$ and $M_{p/m}$, and thus between the permutations $\pi_{m,p}$ and $\tau_{p/m}$. The precise nature of this relationship is, however, a lot less obvious on the level of permutations. It is the purpose of the present note to explore this matter.

We henceforth assume that $m = 1$.

To simplify notation we fix a value of p. We set $\pi := \overline{\pi}_{1,p}$. Note that

$$r = r_p = \frac{(2p - 1) + \sqrt{4p^2 + 1}}{2p}, \quad s = s_p = r_p + \frac{1}{p} = \frac{(2p + 1) + \sqrt{4p^2 + 1}}{2p}.$$

Further notation. If X is an infinite multisubset of $\mathbb{N}$ we write $X = (x_k)$ to denote the elements of X listed in increasing order, thus strictly increasing order when X is an ordinary subset of $\mathbb{N}$. The following four subsets of $\mathbb{N}$ will be of special interest:

$$A_\pi := \{n : \pi(n) > n\} := (a_k),$$
$$B_\pi := \mathbb{N} \backslash A_\pi := (b_k),$$
$$A_\tau := \{n : \tau(n) > n\} := (a_k^*),$$
$$B_\tau := \mathbb{N} \backslash A_\tau := (b_k^*).$$

Note that $b_k = \pi(a_k)$, $b_k^* = \tau(a_k^*)$ for all k. We set

$$\epsilon_k := (b_k - a_k) - (b_k^* - a_k^*) = (b_k - b_k^*) - (a_k - a_k^*).$$

Lemma 1. (i) *For every $n > 0$,*

$$|M_p \cap [1, n]| = |M_{1,p} \cap [1, n]| + \epsilon,$$

where $\epsilon \in \{0, 1, ..., p - 1\}$.

(ii) *$\epsilon_k \in \{0, 1\}$ for all k and if $\epsilon_k = 1$ then $k \not\equiv 0 \pmod{p}$.*

(iii) *$a_{k+1}^* - a_k^* \in \{1, 2\}$ for all $k > 0$ and cannot equal one for any two consecutive values of k.*

(iv) *$b_{k+1}^* - b_k^* \in \{2, 3\}$ for all $k > 0$.*

Proof. (i) and (ii) are easy consequences of the various definitions. (iii) follows from the fact that $r_p \in (3/2, 2)$ and (iv) from the fact that $s_p \in (2, 3)$. $\square$

Main Theorem. *For all $k > 0$, $|a_k - a_k^*| \leq p - 1$.*

Remark. We suspect, but have not yet been able to prove, that $p-1$ is best-possible in this theorem.

Proof of Theorem. The proof is an induction on k, which is most easily phrased as an argument by contradiction. Note that $a_1 = a_1^* = 1$. Suppose the theorem is false and consider the smallest k for which $|a_k^* - a_k| \geq p$. Thus $k > 1$.

`Case I` : $a_k - a_k^* \geq p$. Let $a_k - a_k^* := p' \geq p$. Let b_l be the largest element of B_π in $[1, a_k)$. Then $b_{l-p'+1}^* > a_k^*$ and Lemma 1(iv) implies that $b_l^* - b_l \geq p'$. But Lemma 1(ii) then implies that also $a_l^* - a_l \geq p' \geq p$. Since obviously $l < k$, this contradicts the minimality of k.

`Case II` : $a_k^* - a_k \geq p$. Let $a_k^* - a_k := p' \geq p$. Let b_l^* be the largest element of B_τ in $[1, a_k^*)$. Then $b_{l-p'+1} > a_k$. Lemma 1(iv) implies that $b_{l-p'+1} - b_{l-p'+1}^* \geq p'$ and then Lemma 1(ii) implies that $a_{l-p'+1} - a_{l-p'+1}^* \geq p' - 1$. The only way we can avoid a contradiction already to the minimality of k is if all of the following hold :

 (a) $p' = p$,
 (b) $b_i^* - b_{i-1}^* = 2$ for $i = l, l-1, ..., l-p+2$,
 (c) $l \not\equiv -1 \pmod{p}$ and $\epsilon_{l-p+1} = 1$.

To simplify notation a little, set $j := l - p + 1$. Now $\epsilon_j = 1$ but parts (i) and (ii) of Lemma 1 imply that we must have $\epsilon_{j+t} = 0$ for some $t \in \{1, ..., p-1\}$. Choose the smallest t for which $\epsilon_{j+t} = 0$. Thus

$$b_j^* - a_j^* = b_{j+1}^* - a_{j+1}^* = \cdots = b_{j+t-1}^* - a_{j+t-1}^* = (b_{j+t}^* - a_{j+t}^*) - 1.$$

From (b) it follows that

$$a_{j+t}^* - a_{j+t-1}^* = 1, \quad a_{j+\xi}^* - a_{j+\xi-1}^* = 2, \quad \xi = 1, ..., t-1. \tag{6}$$

Let b_r^* be the largest element of B_τ in $[1, a_j^*)$. Then from (6) it follows that

$$b_{r+t}^* - b_{r+t-1}^* = 3, \quad b_{r+\xi}^* - b_{r+\xi-1}^* = 2, \quad \xi = 2, ..., t-1. \tag{7}$$

Together with Lemma 1(iv) this implies that

$$b_{r+p-1}^* - b_{r+1}^* \geq 2p - 3. \tag{8}$$

But since $a_j^* = a_j - (p-1)$ we have that $b_{r+p-1} < a_j$. Together with (8) this enforces $b_{r+p-1}^* - b_{r+p-1} \geq p$, and then by Lemma 1(ii) we also have $a_{r+p-1}^* - a_{r+p-1} \geq p$. Since it is easily checked that $r + p - 1 < k$, we again have a contradiction to the minimality of k, and the proof of the theorem is complete. □

This theorem implies Conjecture 5.1 of [13]. Recall that the P-positions of $(1, p)$-Wythoff Nim are the pairs $(n - 1, \pi_{1,p}(n) - 1)$ for $n \geq 1$.

Corollary. *With*

$$L = L_p = \frac{s_p}{r_p} = \frac{1 + \sqrt{4p^2 + 1}}{2p}, \quad l = l_p = \frac{1}{L_p},$$

we have that, for every $n \geq 1$,

$$\pi_{1,p}(n) \in \{\lfloor nL \rfloor + \epsilon, \lfloor nl \rfloor + \epsilon : \epsilon \in \{-1, 0, 1, 2\}\}. \tag{9}$$

Proof. We have $\pi_{1,p}(n) = n$ for $n = 1, ..., p$, and one checks that (11) thus holds for these n. For $n > p$ we have by (5) that

$$\pi_{1,p}(n) = \pi(n - p) + p, \tag{10}$$

where $\pi = \overline{\pi}_{1,p}$. There are two cases to consider, according as to whether $n - p \in A_\pi$ or B_π. We will show in the former case that $\pi_{1,p}(n) = \lfloor nL \rfloor + \epsilon$ for some $\epsilon \in \{-1, 0, 1, 2\}$. The proof in the latter case is similar and will be omitted.

So suppose $n - p \in A_\pi$, say $n - p = a_k$. Then

$$\pi(a_k) = b_k = a_k + (b_k^* - a_k^*) + \epsilon_k. \tag{11}$$

Moreover $a_k^* = \lfloor kr_p \rfloor$ and $b_k^* = \lfloor ks_p \rfloor$, from which it is easy to check that

$$b_k^* = a_k^* L + \delta, \quad \text{where } \delta \in (-1, 1).$$

Substituting into (11) and rewriting slightly, we find that

$$\pi(a_k) = a_k L + (a_k^* - a_k)(L - 1) + \delta + \epsilon_k,$$

and hence by (10) that $\pi_{1,p}(n) = nL + \gamma$ where

$$\gamma = (a_k^* - a_k - p)(L - 1) + \delta + \epsilon_k.$$

By Lemma 1, $\epsilon_k \in \{0, 1\}$. By the Main Theorem, $|a_k^* - a_k| \leq p - 1$. It is easy to check that $(2p-1)(L-1) < 1$. Hence $\gamma \in (-2, 2)$, from which it follows immediately that $\pi_{1,p}(n) - \lfloor nL \rfloor \in \{-1, 0, 1, 2\}$. This completes the proof. $\square$

Remark. As stated in Section 5 of [13], computer calculations seem to suggest that, in fact, (9) holds with just $\epsilon \in \{0, 1\}$. So once again, the results presented here may be possible to improve upon.

Acknowledgements At the Integers 2007 Conference when I introduced Imitation Nim to Aviezri Fraenkel I believe he quickly responded *"Limitation Nim"*. As this name emphasises another important aspect of the game, I would like to propose it for the "dual" of Wythoff's original game, that is whenever one wants to emphasise that no imitation is allowed. I would also like to thank A. Fraenkel for contributing with some references and the anonymous referee for the references and the interesting discussion on strategies in a wide and broad sense (footnote 2 on page 6).

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