The $\star$-operator and invariant subtraction games

Urban Larsson, Chalmers and University of Gothenburg, BIRS CGT-conference 2011.

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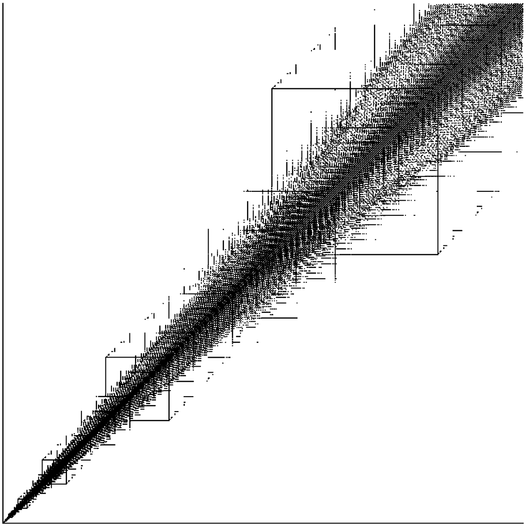

Invariant Subtraction Games, each move is available inside the whole board

Let (the poset) $\mathcal{B} = \mathbb{N}_0^k$, $k \in \mathbb{N}$, denote the 'game board'. ($\mathbb{N} = \{1, 2, \ldots\}, \mathbb{N}_0 = \mathbb{N} \cup \{0\}$), so that every position is a k-tuple $(x_1, x_2, \ldots, x_k)$. Hence we have a natural order of the positions. Then the impartial game G is an invariant subtraction game (game) if there is a set $\mathcal{M} = \mathcal{M}(G) \subseteq \mathcal{B} \setminus \{\mathbf{0}\}$ such that for all $\boldsymbol{r} \in \mathcal{M}$ and all $\boldsymbol{x} \in \mathcal{B}$ such that $\boldsymbol{x} - \boldsymbol{r} \succeq \mathbf{0}$ ("vector subtraction"),

$$\boldsymbol{x} \rightarrow \boldsymbol{x} - \boldsymbol{r}$$

is a legal move in G.

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$$\boldsymbol{x} \to \boldsymbol{x} - \boldsymbol{r}$$

is a legal move in G. The notion of invariant games was introduced by Duchêne and Rigo, 2009 (together with an interesting conjecture).

P- and N-positions

Given an invariant subtraction game, there are no 'cyclic' moves. In normal play, the player who moves last wins.

- A position is P if all options are N. Otherwise it is N.

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- A position is P if all options are N. Otherwise it is N.
- The first player wins if and only if the position is N.
- $\mathcal{P}(G)$ (resp. $\mathcal{N}(G)$) is the collection of P- (resp. N-) positions of a game G.

2-pile Nim, an invariant subtraction game

$\mathcal{M}(\text{Nim}) = \{(0,i),(i,0) \mid i \in \mathbb{N}\}$. $\mathcal{P}(\text{Nim}) = \{(i,i) \mid i \in \mathbb{N}_0\}$.

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- Pile-heights decreases, a subtraction game.
- Invariance of moves.

The definition of $G^\star$

If G is a (not necessarily invariant) game, then we can define an invariant game $G^\star$ on the same game board by setting

$$\mathcal{M}(G^\star) := \mathcal{P}(G) \backslash \{\mathbf{0}\}.$$

Sequences of games

When does a limit game exist?

The sequence $(G(n))_{n\in\mathbb{N}}$ of invariant subtraction games converges (to a limit game $H = \lim_{n\in\mathbb{N}} G(n)$) if, for all $\boldsymbol{x} \in \mathcal{B}$ and for all $\boldsymbol{m} \preceq \boldsymbol{x}$, there is an $N = N(\boldsymbol{x})$ such that, for all $n \geq N$, $\boldsymbol{m} \in \mathcal{M}(G(n))$ if and only if $\boldsymbol{m} \in \mathcal{M}(G(N))$.

A limit game exists

Repeated applications of $\star$

Let G denote an invariant subtraction game. Then G^n denotes the resulting game after n applications of the $\star$-operator on G. That is, $G^0 = G,\ G^1 = G^\star,\ G^2 = G^{\star\star} = (G^\star)^\star, \ldots$.

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Theorem (Main Theorem)

The sequence $\left(G^{2n}\right)$ *converges.*

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Theorem (Main Theorem)

The sequence $\left(G^{2n}\right)$ *converges.*

We sketch a proof, but first a few examples...

'$\star$' as in 'dual'

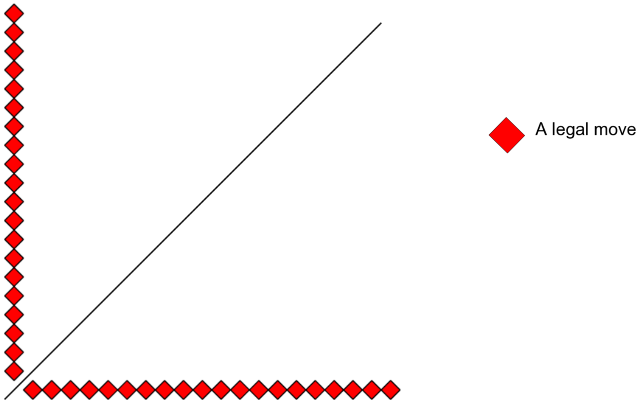
If a game G satisfies $G = G^{\star\star}$, then we call $G^\star$ its 'dual' (game).

'$\star$' as in 'dual'

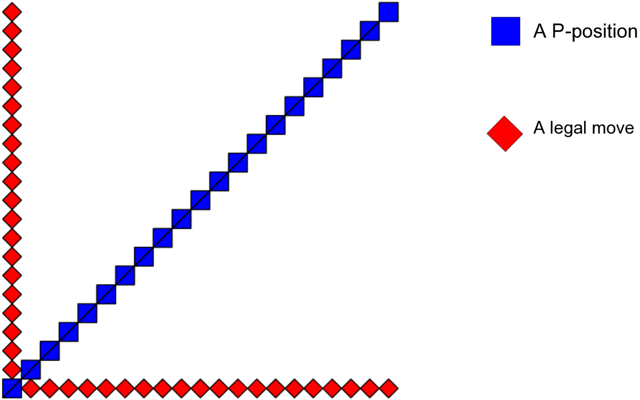
If a game G satisfies $G = G^{\star\star}$, then we call $G^{\star}$ its 'dual' (game).

A generic game does not have a dual, but particular instances (of game families) do... For such examples our Main Theorem 'trivially' hold.

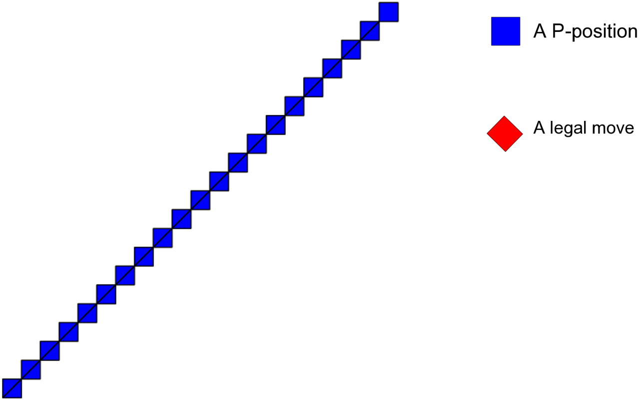
Example: Nim



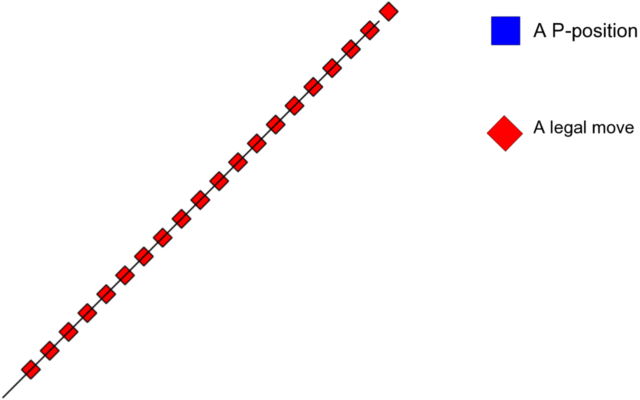
Nim and its P-positions



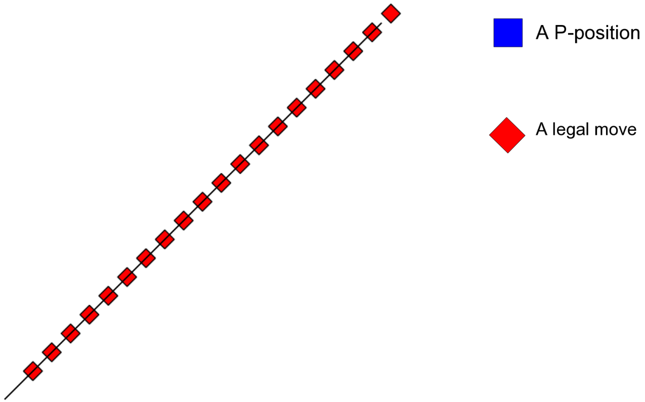
The P-positions of Nim



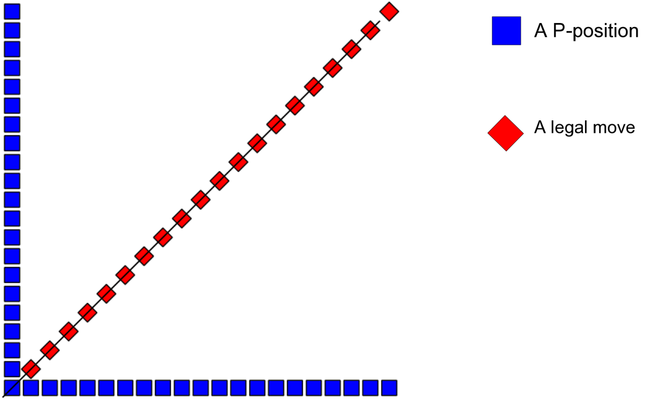
The moves of Nim ⋆



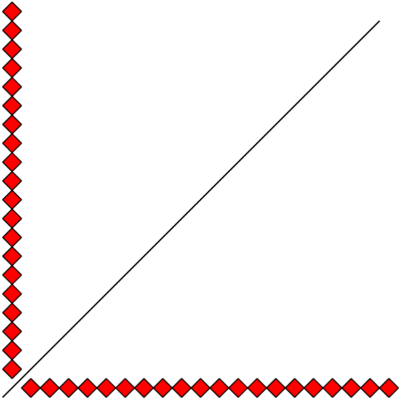
Wait, what are the P-positions of Nim *?

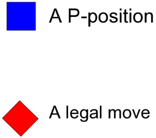


The moves and P-positions of Nim ⋆

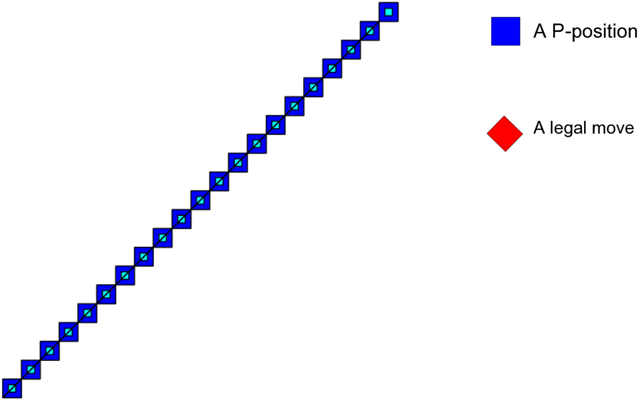


The game of (Nim ⋆) ⋆ equals Nim

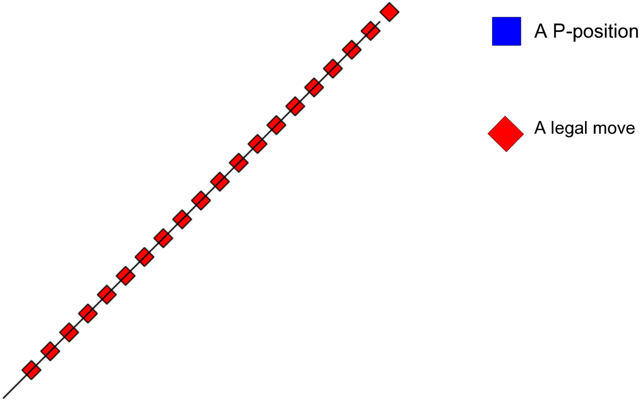




Hence, the P-positions are identical



Thus, Nim * may be regarded as the 'dual' of Nim



Nim has a dual game

This result holds also for general k-pile Nim .

Theorem

(k-pile Nim)** = k-pile Nim

We give the very short proof in the final slide.

Complementary sequences

A pair of sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of positive integers is complementary if $\{x_n\} \cup \{y_n\} = \mathbb{N}$ and $\{x_n\} \cap \{y_n\} = \emptyset$.

Invariant games on complementary sequences.

Let $\mathcal{T}(G)$ denote the set of terminal P-positions of a game G.

Theorem

Let $a = (a_i)_{i\in\mathbb{N}}$ and $b = (b_i)_{i\in\mathbb{N}}$ denote complementary sequences of positive integers, a increasing, and for all i, $a_i < b_i$. Define G by $\mathcal{M}(G) = \{\{a_i, b_i\} \mid i \in \mathbb{N}\}$. Then,

(i) *$(x, y) \in \mathcal{P}(G) \setminus \mathcal{T}(G)$ implies that there is an $i \in \mathbb{N}$ such that $x = a_i$ or $y = a_i$.*

(ii) *if b is increasing, then $(x, y) \in \mathcal{P}(G) \setminus \mathcal{T}(G)$ implies that there are $i, j \in \mathbb{N}$ such that $x = a_i$ and $y = a_j$.*

(iii) *if b_i/a_i is bounded by some constant, say $C \in \mathbb{R}$, then $(x, y) \in \mathcal{P}(G) \setminus \mathcal{T}(G)$ (with $x \leq y$) implies that $y/x \leq C$.*

Complementary Beatty sequences

The Rayleigh/Beatty theorem (1894/1927)

We say that the ordered pair (α, β) is a Beatty pair if $\alpha < \beta$ are positive irrationals with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Observation, then $1 < \alpha < 2 < \beta$.

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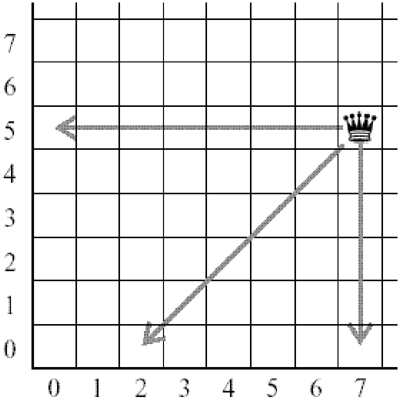
We say that the ordered pair (α, β) is a Beatty pair if $\alpha < \beta$ are positive irrationals with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Observation, then $1 < \alpha < 2 < \beta$. Let $\alpha < \beta$ denote positive real numbers. Then the sequences $(\lfloor n\alpha \rfloor)_{n \in \mathbb{N}}$ and $(\lfloor n\beta \rfloor)_{n \in \mathbb{N}}$ are complementary if and only if (α, β) is a Beatty pair.

Wythoff Nim

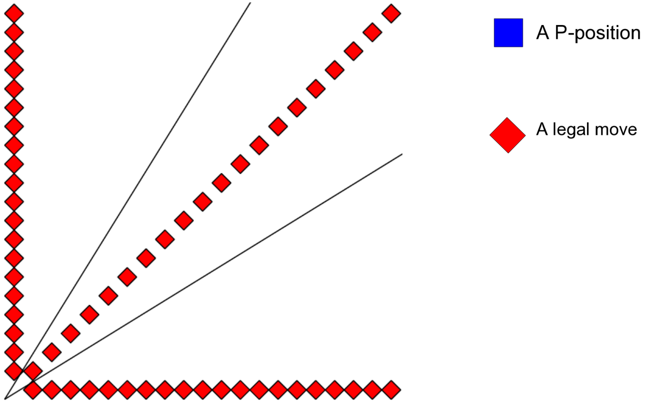
A golden instance of Beatty pairs

Let $\phi := \frac{1+\sqrt{5}}{2}$ denote the Golden ratio. Our next example concerns the Beatty pair (ϕ, ϕ^2).

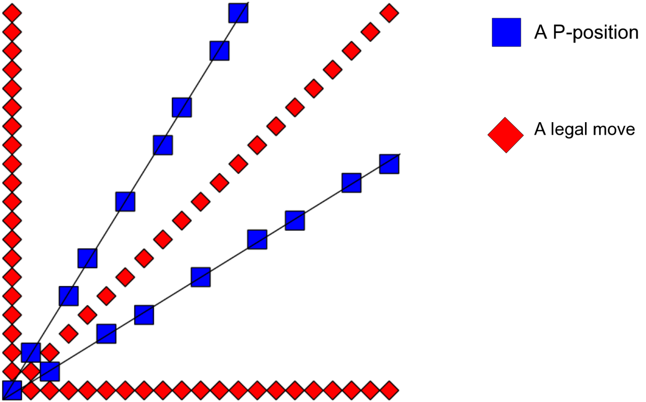
Wythoff Nim (1907), 'Corner the Queen'



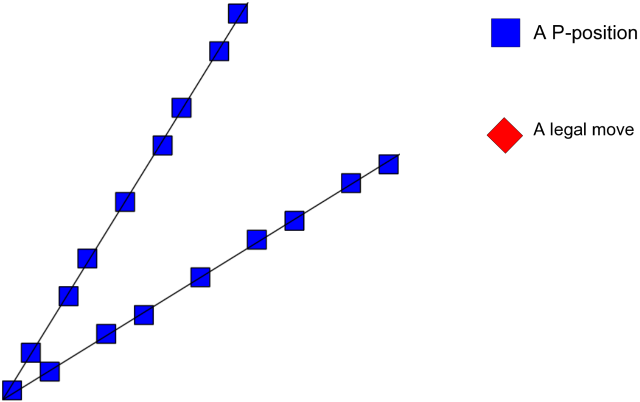
Wythoff Nim and the lines ϕx and $\frac{x}{\phi}$.



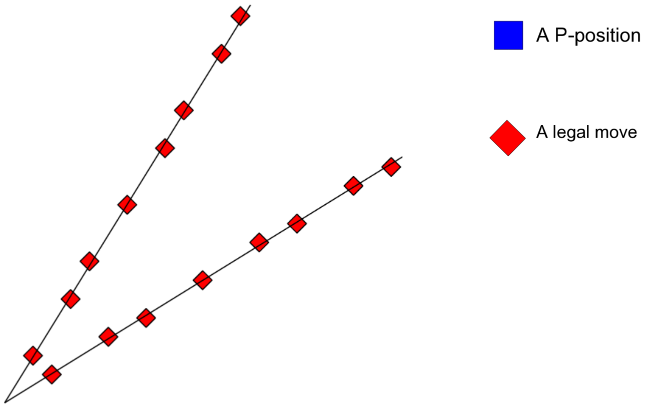
The moves and P-positions of Wythoff Nim



The P-positions of Wythoff Nim, $(\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor)$, $n \in \mathbb{N}_0$



The initial moves of (Wythoff Nim)*



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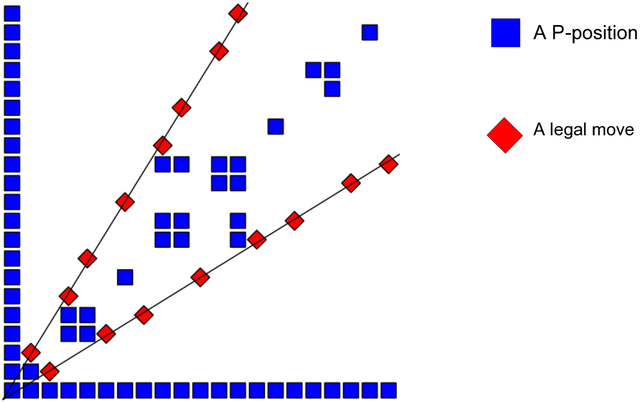

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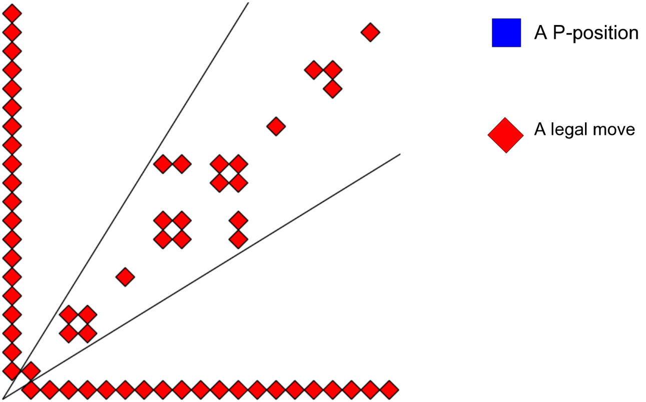
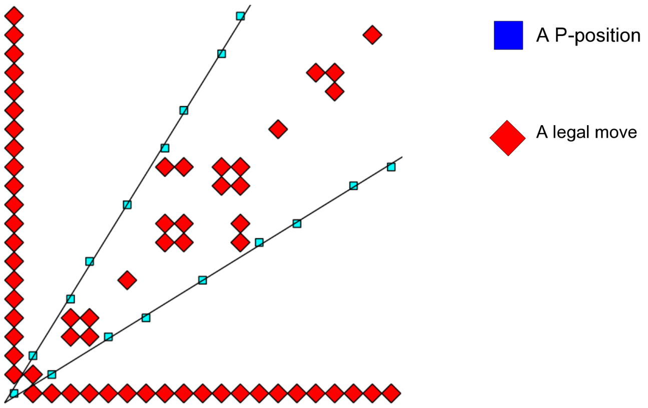
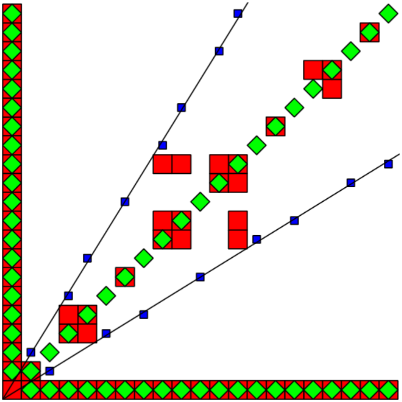


The initial moves and *P*-positions of ((Wythoff Nim)*)*

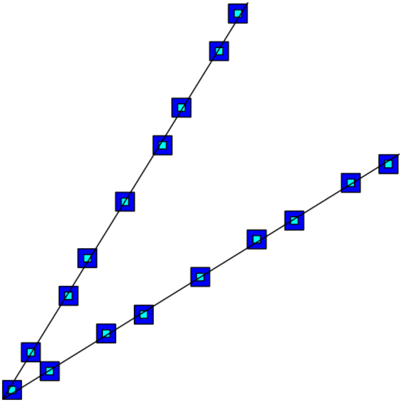


Wythoff Nim does not have a dual game, but (Wythoff Nim)* has.

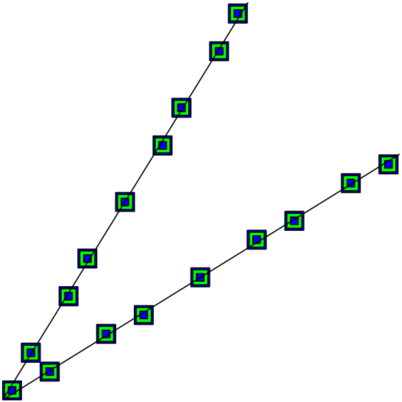
Wythoff Nim $\neq$ ((Wythoff Nim)*)*



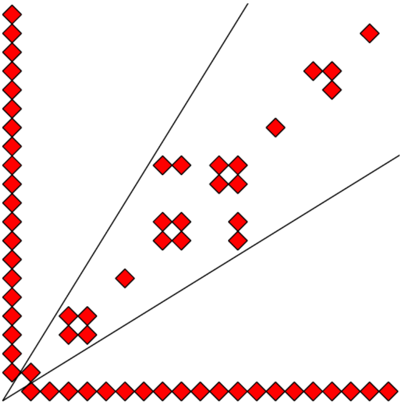
$\mathcal{P}(\text{Wythoff Nim}) = \mathcal{P}\left(((\text{Wythoff Nim})^{\star})^{\star}\right)$

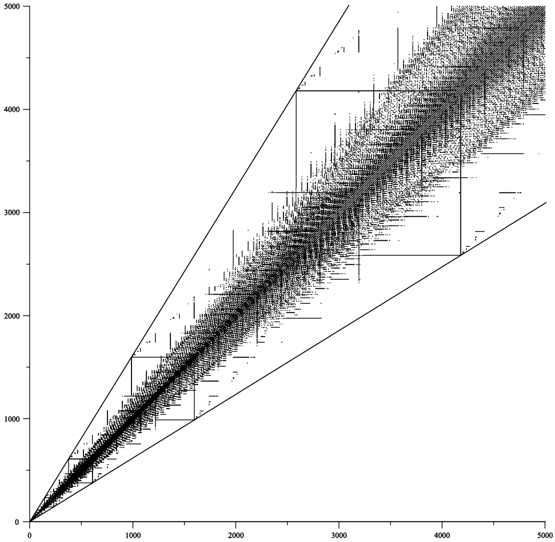


Identical P-positions of (Wythoff Nim)^{2k*}, $k \in \mathbb{N}_0$



Identical moves of (Wythoff Nim)^{2k*}, $k \in \mathbb{N}$. What are they? Is there a 'closed formula'?

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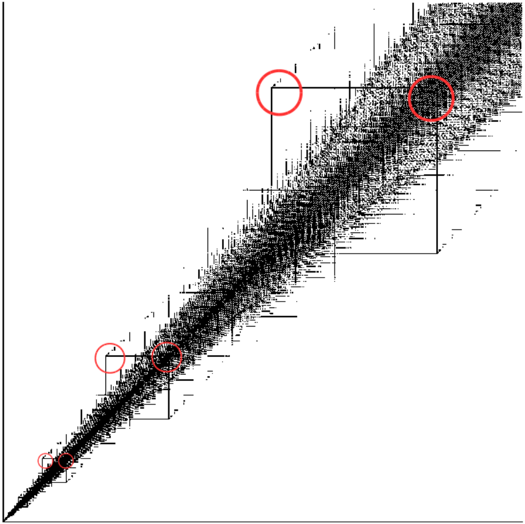

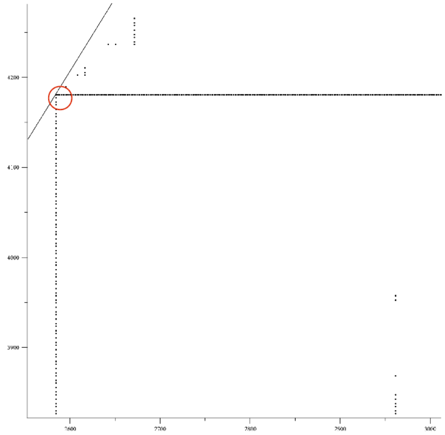
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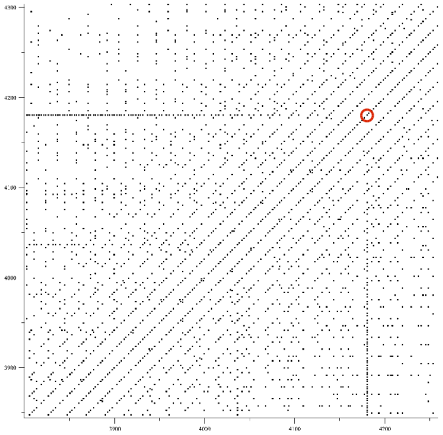
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Theorem

Let $F_0 = F_1 = 1, F_n = F_{n-1} + F_{n-2}$, $(n \geq 2)$ denote the sequence of Fibonacci numbers. Then, for all n, provided both coordinates are positive, the following positions (and its symmetric counterparts) belong to $\mathcal{P}$(Wythoff Nim*)

(i) $(F_{2n-1}, F_{2n} - 1)$,

(ii) $(F_{2n-1}, F_{2n} - 4)$,

(iii) $(F_{2n-1}, F_{2n} - 9)$,

(iv) $(F_{2n-1} + 1, F_{2n} - 1)$,

(v) $(F_{2n-1} + 3, F_{2n} - 1)$,

(vi) $(F_{2n-1} + 4, F_{2n} - 1)$,

(vii) $(F_{2n-1} + 6, F_{2n} - 1)$, *and*

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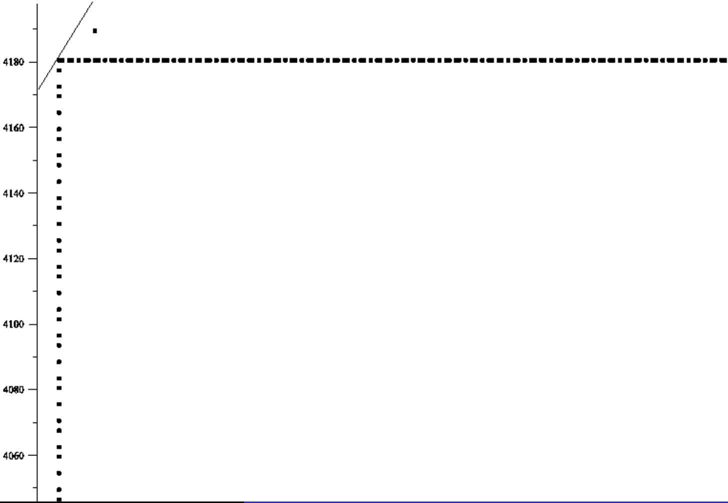
(ii) $(F_{2n-1}, F_{2n} - 4)$,
(iii) $(F_{2n-1}, F_{2n} - 9)$,
(iv) $(F_{2n-1} + 1, F_{2n} - 1)$,
(v) $(F_{2n-1} + 3, F_{2n} - 1)$,
(vi) $(F_{2n-1} + 4, F_{2n} - 1)$,
(vii) $(F_{2n-1} + 6, F_{2n} - 1)$, *and*
(viii) $(F_{2n} - 1, F_{2n} - 1)$.

A ‘limb’ conjecture



A 'limb' conjecture

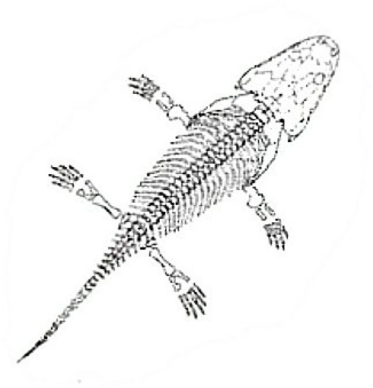
Let $A_i = \lfloor \phi i \rfloor$ and $B_i = A_i + i$, $i \in \mathbb{N}$.

Conjecture

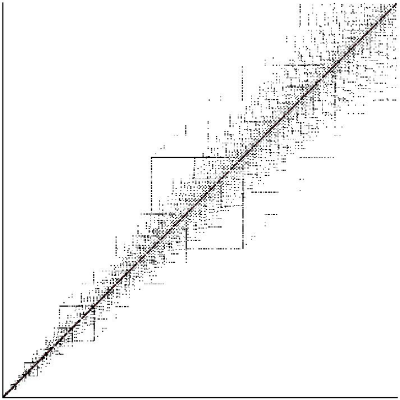
For all $n \geq 3$ and all i such that

- $A_i + B_i \leq F_{2n-4}$, *the position* $(F_{2n-1}, F_{2n} - 1 - A_i - B_i)$ *is* P.
- $A_i \leq F_{2n-4}$, *the position* $(F_{2n-1} + A_i, F_{2n} - 1)$ *is* P.

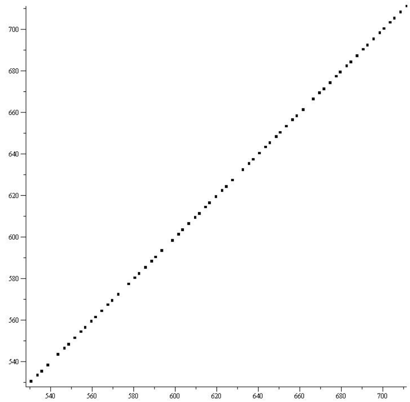
A vertebrate



A 'spine' conjecture



A 'spine' pattern?



A 'spine' conjecture

Conjecture

Let

$S_1 := \{3, 8, 11, 21, 32\}$,
$S_2 := \{129, 362\}$,
$S_3 := \{x \in \mathbb{N} \setminus \{19\} \mid$ *The Zeckendorf coding of x ends in 101001*$\}$,
$S_4 := \{x \in \mathbb{N} \mid$ *The Zeckendorf coding of x ends in 1*$\}$.

Then, the position (i, i) *belongs to* $\mathcal{P}(W^\star)$ *if* i *belongs to* $(S_1 \cup S_4) \setminus (S_2 \cup S_3)$. *It belongs to* $\mathcal{N}(W^\star)$ *if* i *belongs to* $\mathbb{N} \setminus (S_1 \cup S_3 \cup S_4)$.

A 'spine' conjecture

Conjecture

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$$S_1 := \{3, 8, 11, 21, 32\},$$
$$S_2 := \{129, 362\},$$
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$$S_4 := \{x \in \mathbb{N} \mid \textit{The Zeckendorf coding of } x \textit{ ends in } 1\}.$$

Then, the position (i, i) *belongs to* $\mathcal{P}(W^\star)$ *if* i *belongs to* $(S_1 \cup S_4) \setminus (S_2 \cup S_3)$*. It belongs to* $\mathcal{N}(W^\star)$ *if* i *belongs to* $\mathbb{N} \setminus (S_1 \cup S_3 \cup S_4)$.

No conjecture on the 'torso'...

The Duchêne-Rigo conjecture on invariant games

Conjecture (Duchêne-Rigo, 2009)

Suppose that (α, β) *is a Beatty pair. Then there exists an invariant removal (subtraction) game with its set of P-positions identical to* $\{(\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor), (\lfloor n\beta \rfloor, \lfloor n\alpha \rfloor) \mid n \in \mathbb{N}_0\}$.

t-superadditive-complementarity, t-SAC

The Duchêne-Rigo conjecture was formulated for complementary Beatty sequences. We (joint with Hegarty and Fraenkel, 2010) have proved it in a somewhat more general setting.

t-superadditive-complementarity, t-SAC

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Let $t \in \mathbb{N}$. We say that a sequence $(X_n)_{n\in\mathbb{N}_0}$ of non-negative integers is t-superadditive if, for all $m, n \in \mathbb{N}_0$,

$$X_m + X_n \leq X_{m+n}$$

t-superadditive-complementarity, t-SAC

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$$X_m + X_n \leq X_{m+n} < X_m + X_n + t.$$

t-superadditive-complementarity, t-SAC

Let $a = (a_n)_{n\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$ be sequences of positive integers and define $a_0 = b_0 = 0$. The set $\{(a_n, b_n) \mid n \in \mathbb{N}_0\}$ of ordered pairs is t-superadditive-complementary, abbreviated t-SAC, if the following criteria are satisfied:

- $a_1 = 1$,

t-superadditive-complementarity, t-SAC

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- $a_1 = 1$,
- a and b are complementary sequences,

t-superadditive-complementarity, t-SAC

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- b is t-superadditive.

Another ‘duality’ theorem

Theorem

Define G by setting $\mathcal{M}(G) := \{(a_n, b_n), (b_n, a_n) \mid n \in \mathbb{N}\}$, where $\{(a_n, b_n) \mid n \in \mathbb{N}_0\}$ is b_1-SAC. Then

$$\mathcal{P}(G^\star) = \mathcal{M}(G) \cup \{\mathbf{0}\}$$

and hence

$$(G^\star)^\star = G.$$

As a consequence the Duchêne-Rigo conjecture holds

Observation

Any homogeneous Beatty sequence is 2-superadditive. Hence, if a and b is a pair of complementary homogeneous Beatty sequences, then the set $\{(a_n, b_n) \mid n \in \mathbb{N}_0\}$ is 2-SAC, hence b_1-SAC.

Corollary

Let (α, β) be a Beatty pair. Then there exists an invariant subtraction game G such that $\mathcal{P}(G) = \{(\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor), (\lfloor n\beta \rfloor, \lfloor n\alpha \rfloor) \mid n \in \mathbb{N}_0\}$.

Other complementary pairs of sequences

What if the *b*-sequence does not increase?

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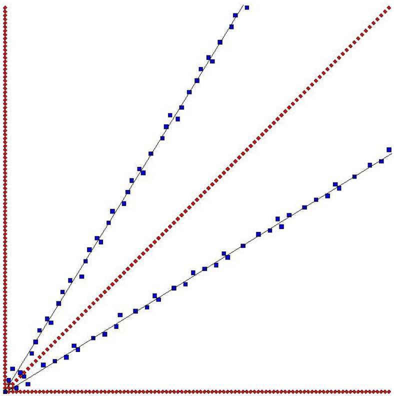
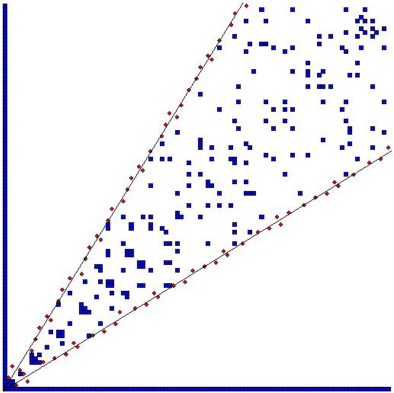
- Maharaja Nim: moves as in Wythoff Nim, but also $(1,2)$ and $(2,1)$,
- $(1,2)$GDWN: moves as in Maharaja Nim, but also $(t,2t)$ or $(2t,t)$, $t \in \mathbb{N}$.

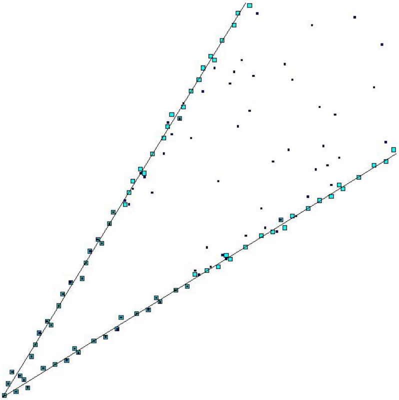
Maharaja Nim



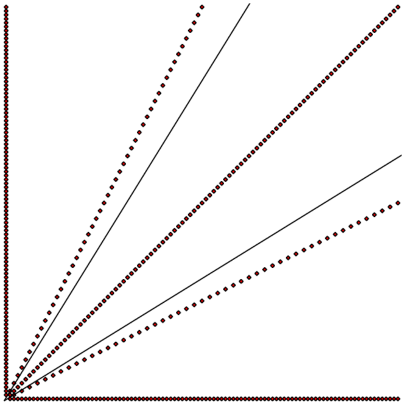
Maharaja Nim*: the b-sequence does not increase...



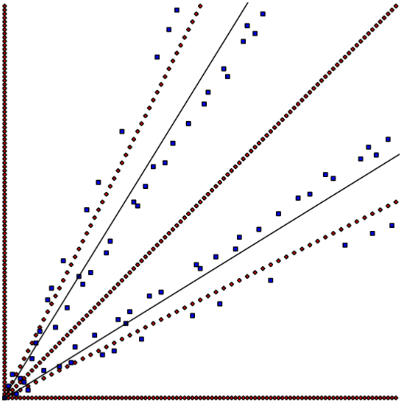
$\mathcal{P}$(Maharaja) and $\mathcal{P}$(Maharaja**) are disjoint



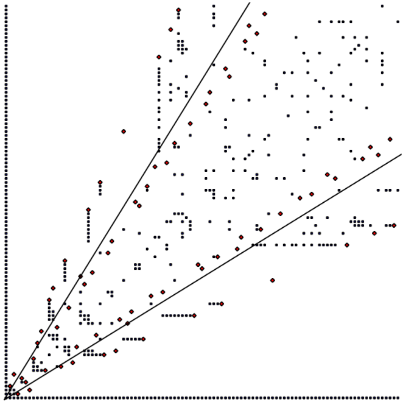
The moves of (1, 2)GDWN and the lines ϕx and $\frac{x}{\phi}$



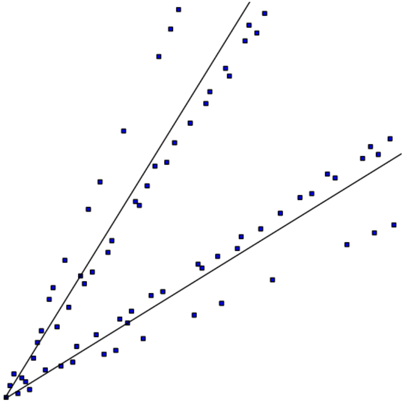
$(1, 2)$GDWN: The P-positions seem to "split"



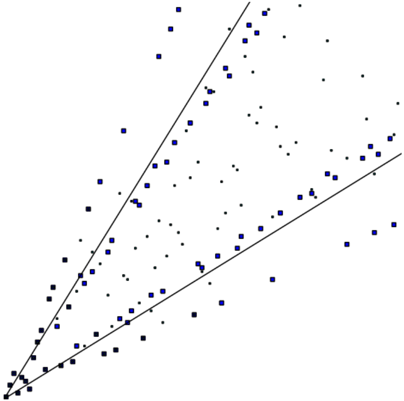
Moves and P-positions of $(1,2)\mathrm{GDWN}^{\star}$



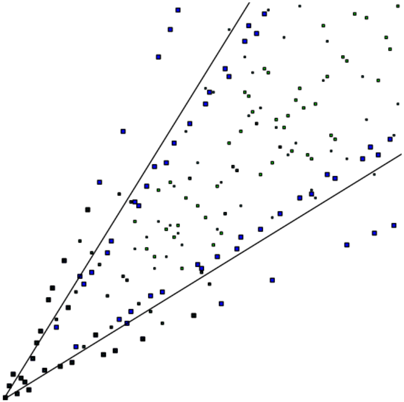
$\mathcal{P}((1,2)\text{GDWN})$



$\mathcal{P}((1,2)\mathrm{GDWN}^{2\star})$



$\mathcal{P}((1,2)\mathrm{GDWN}^{4\star})$



Open questions about convergence

Is there a k such that for all $k \le l \in \mathbb{N}$,

- $\mathcal{P}((\text{Maharaja Nim})^{2l\star}) = \mathcal{P}((\text{Maharaja Nim})^{2k\star})$?

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- In general, are there games such that the answer is yes to the above question for each $k \in \mathbb{N}$?

Permutation and Involution games

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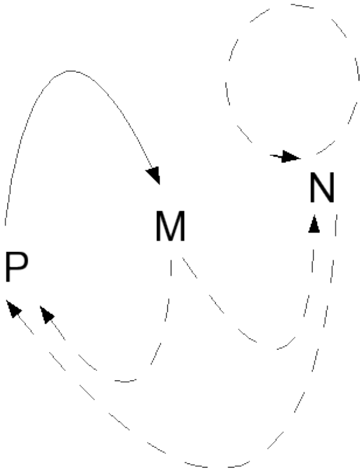
- Remark: Symmetry of moves is not 'necessary for closure'...
- The terminology of the next theorem is demonstrated in a Java-applet (available on-line at my home page) by Gunnar Stenlund, student mathematician at Chalmers.

Permutation games and finitive Nim extensions

Theorem

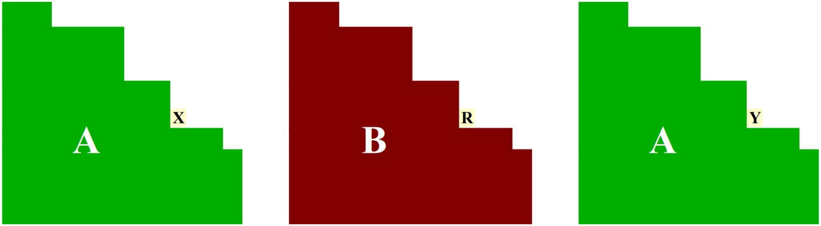
- *$\star\star$ of any game with at least one position in each row and column is a permutation game.*
- *Hence, the family of permutation (involution) games is closed under $\star\star$,*
- *and so is the family of (symmetric) finitive Nim extensions.*
- *$\star$ of a permutation (involution) game is a (symmetric) finitive Nim extension.*
- *$\star$ of a (symmetric) finitive Nim extension is a (involution) permutation game.*
- *Hence, by the Main Theorem, the limit game of a permutation (involution) game is also a permutation (involution) game.*

A proof of the Main Theorem, the ⋆-operator



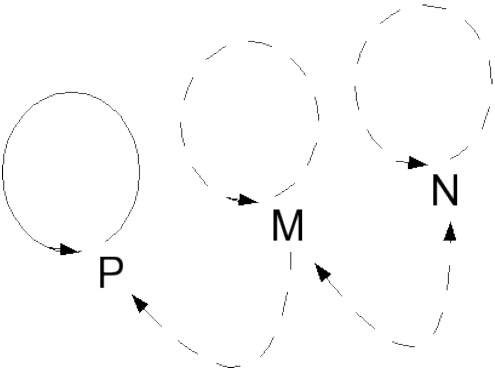
A proof of the Main Theorem, the ⋆-operator

Suppose that some configuration of moves strictly below a certain position remains fixed under the operation of ⋆⋆. Then the 'total configuration' of Ms, Ps, and Ns (A in the figure) remains fixed on the corresponding 'lower-left' board $\subset \mathcal{B}$. How does the status X of this 'least position not in A' shift?



A proof of the Main Theorem, the ⋆⋆-operator

Given a fixed lower-left configuration, the shift of status of some least position not in this configuration under the operation of ⋆⋆:



The convergence

Hence, for a fixed lower-left configuration A the status of some least position not in A, under a successive application the $\star$-operator follows one out of a total of 6 patterns:

$$N \to N \to N,$$
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$$M \to N \to P \to M \to P, \text{ or}$$
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... is proved

We are done. □

Ornament games and pairs of rational Beatty sequences

The family of so-called 'Ornament games' defined via complementary (inhomogeneous) Beatty sequences with rational moduli (Fraenkel 1969) and the ⋆-operator provide interesting questions on periodicity and duality. Example, the 'the Mouse trap' is an invariant game with the same P-positions as the variant 'Mouse game' (Fraenkel GONC 4), namely $\{(\lfloor 3n/2 \rfloor, 3n-1), (3n-1, \lfloor 3n/2 \rfloor) \mid n \in \mathbb{N}\}$. There are precisely three Ornament games with modulus 3.

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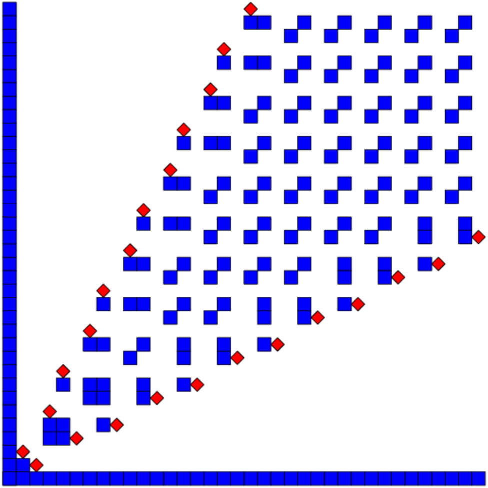
b-sequence has modulus 3, the 'Mouse trap'



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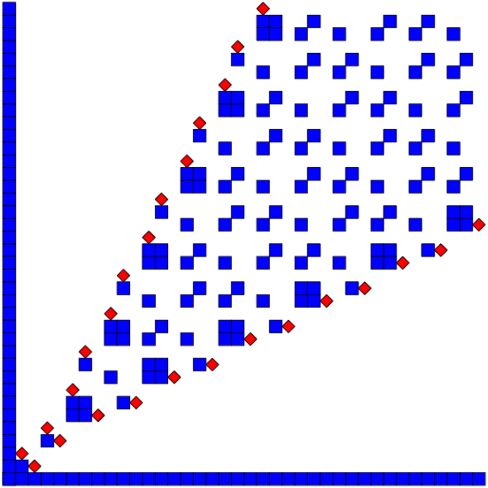

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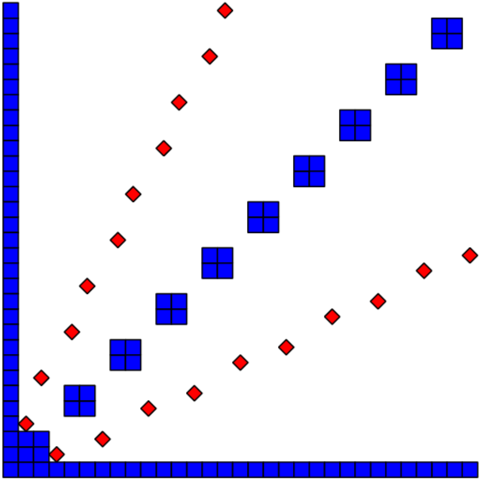

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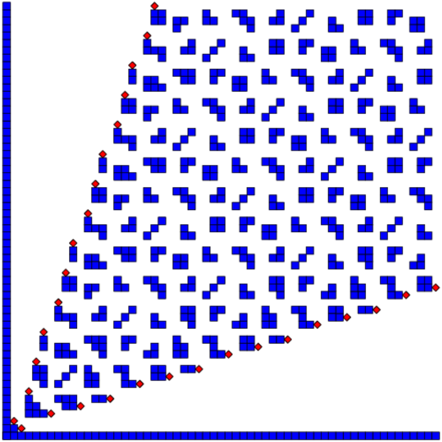

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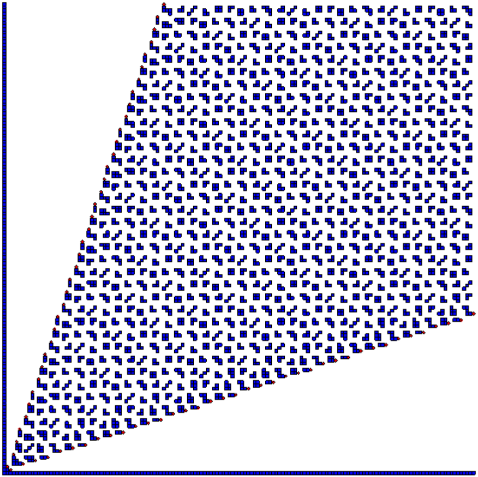

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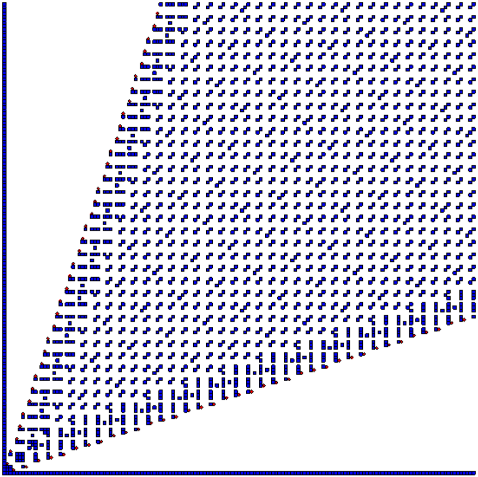

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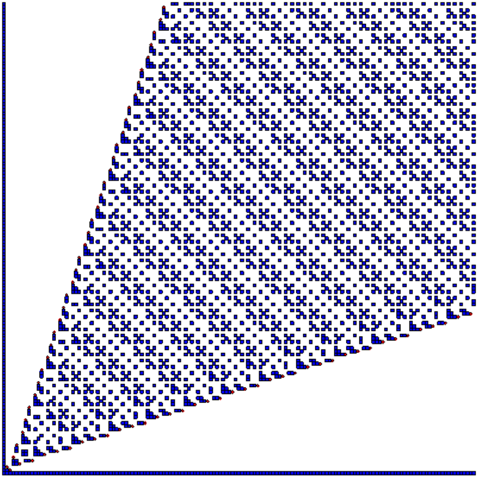
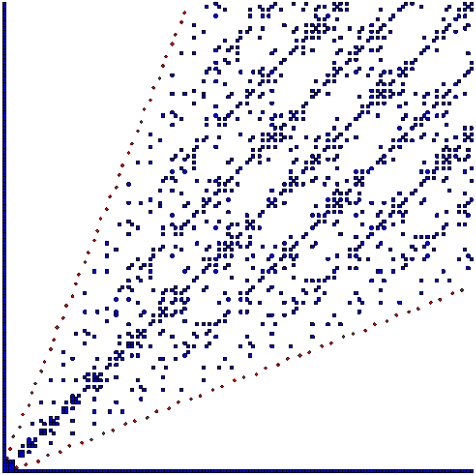


b-sequence has modulus 7



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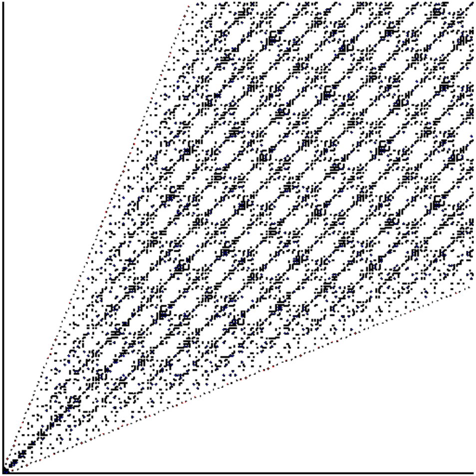

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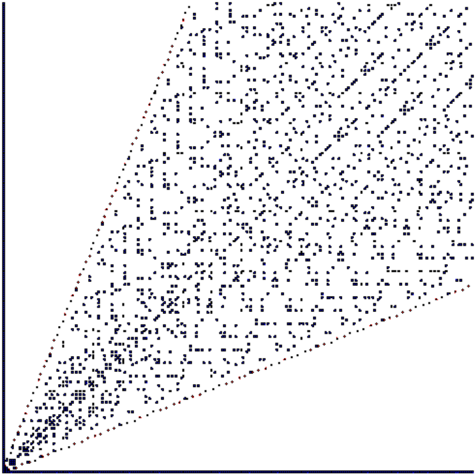

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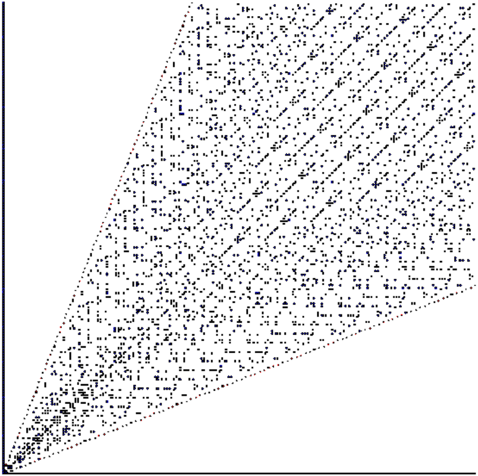

Nim on several piles, Nim^k

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Let $k \in N$. Then $Nim^k = (Nim^k)^{\star\star}$.

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