From heaps of matches to the limits of computability

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CANT 2011

May 24, 2011

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Place a heap of tokens on a table

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The move options

Figure: Rules: remove "one", "two" or "three" tokens.

How to win

Figure: The Previous player's safe positions are divisible by four.

A winning move

Figure: P-positions divisible by four.

The new move options are loosing

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- the set of P-positions will ultimately become periodic.
- The game cannot embrace any mysteries.

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- a position can be regarded as a d-dimensional vector $x = (x_1, \dots, x_d)$ of non-negative integers.
- The rules of the game are encoded by a finite set $\mathcal{M}$ of integer vectors that specify the permitted moves.
- If $(m_1, \dots, m_d) \in \mathcal{M}$, then from position $(x_1, \dots, x_d)$, a player can move to position $(x_1 + m_1, \dots, x_d + m_d)$, provided all of the numbers $x_1 + m_1, \dots, x_d + m_d$ are non-negative.

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- If some moves involve adding tokens to heaps, the game does not necessarily have to terminate.
- In each move, the total number of tokens decreases.

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- For any given position $x = (x_1, \ldots, x_d)$ we can determine recursively whether or not x is a P-position by first computing the status of all positions with fewer tokens.
- The position x is in $\mathcal{P}(\mathcal{M})$ iff there is no move from x to a position in $\mathcal{P}(\mathcal{M})$.

Patterns and P-positions

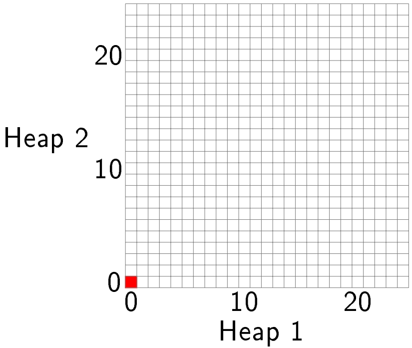

Figure: The P-positions of the game $\mathcal{M} = \{(-1,-3),(-2,1)\}$.

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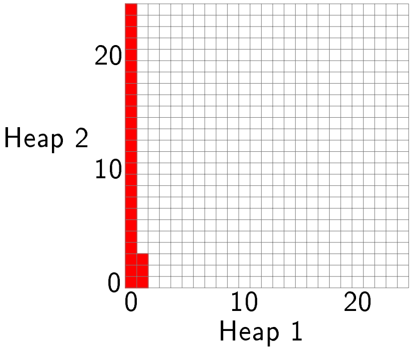

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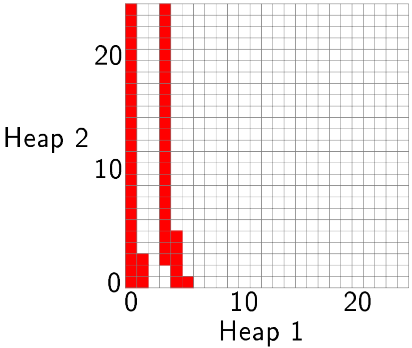

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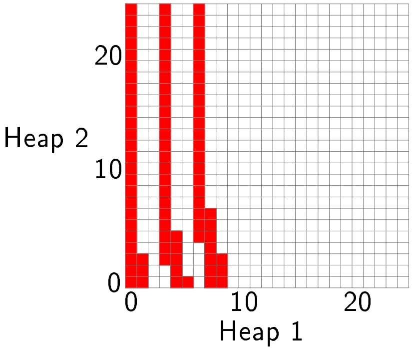

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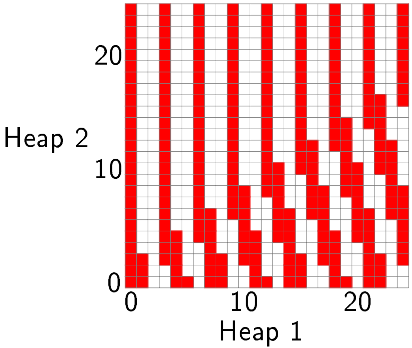

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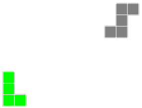

Figure: A pattern that occurs in $\mathcal{P}$ an one that does not

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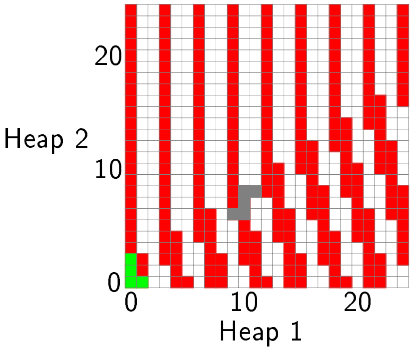
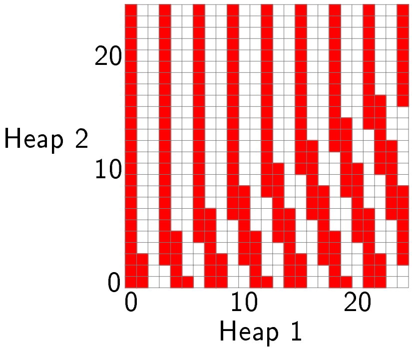


Figure: Pattern occurrence

Patterns and P-positions



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Undecidable games

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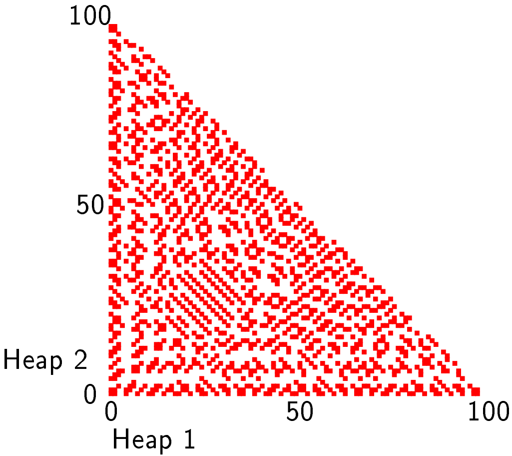
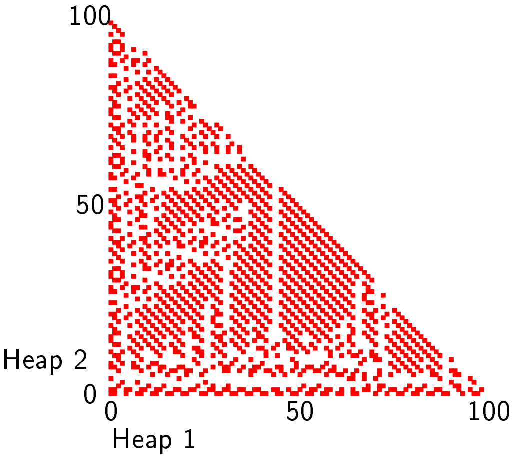
- If we cannot answer questions of pattern occurrence, we cannot claim to have full understanding of the set of P-positions of a game.

$\mathcal{M} = \{(0,-2), (-2,0), (2,-3), (-3,2), (-3,-3)\}$



$$\mathcal{M} = \{(0,-2),(-2,0),(2,-3),(-3,2),(-5,4),(-5,-2),(-4,-3),(-1,-4)\}$$



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Theorem

There is no algorithm that, given as input the set of moves $\mathcal{M}$ and a finite pattern, decides whether the given pattern occurs in $P(\mathcal{M})$.

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- Two states (0 and 1), and the state of cell i at time t is denoted by $x_{t,i}$.
- The starting configuration is ...000111... (the first '1' at position 0).

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- We require that $f(0,\ldots,0) = 0$, so that $x_{t,i} = 0$ whenever $i < 0$.

Example

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$$f(x, y) = x \oplus y,$$

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- The leftmost column represents $x_{t,0}$.

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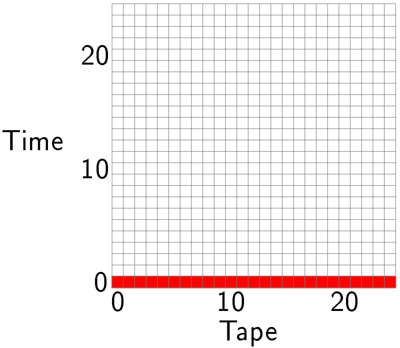

Figure: The CA given by $f(x, y) = x \oplus y$ (Wolfram's rule 90).

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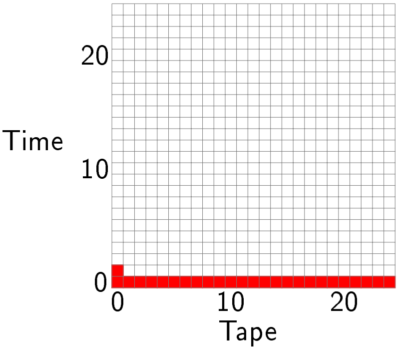

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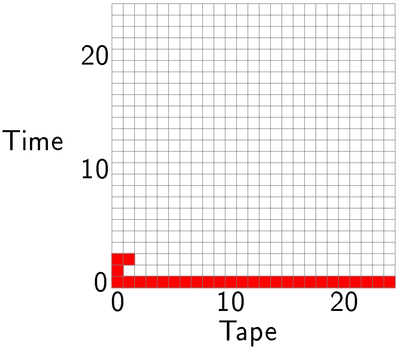

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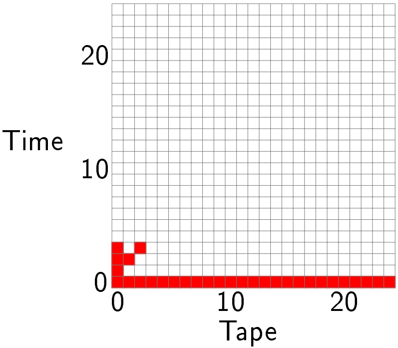

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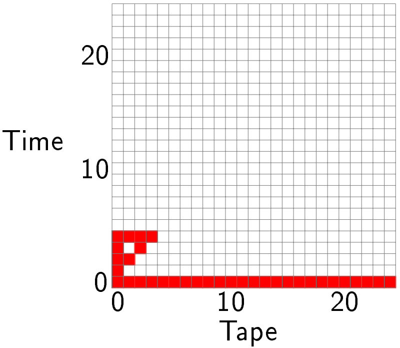

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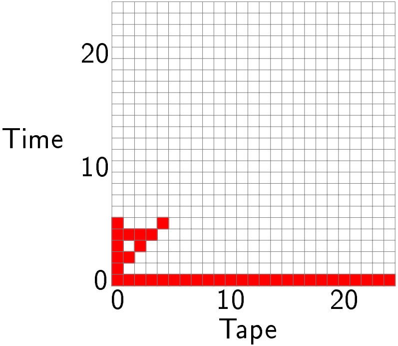

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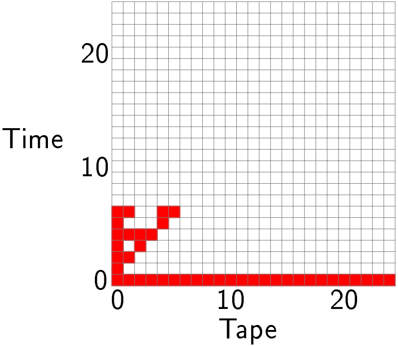

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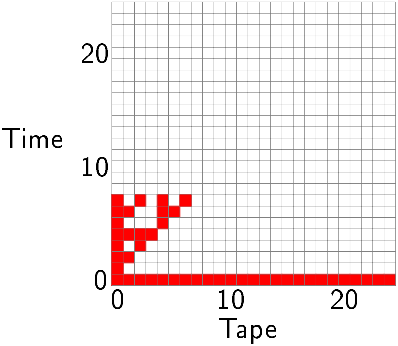

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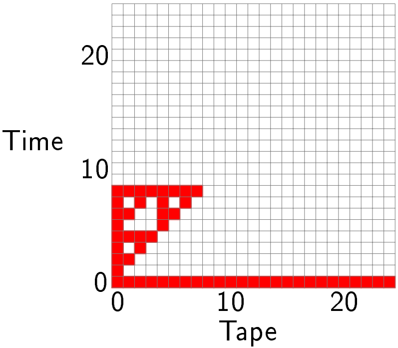

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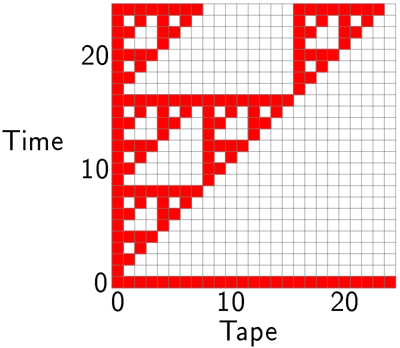
The pattern is Pascal's triangle modulo 2



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A well-known undecidability result

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Theorem

There is no algorithm that takes input n, f, and a bit-string s of length n, and answers whether or not s ever occurs in the CA given by f.

CA: Rule 110

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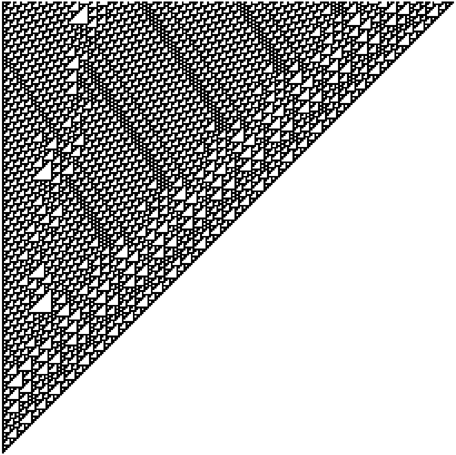
- Update rule: "0s are changed to 1s at all positions where the value to the right is a 1, while 1s are changed to 0s at all positions where the values to the left and right are both 1".

CA, rule 110, time moves upwards



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- The Time heap: we insert some space, k, to allow for the rules of the game to "compute" an arbitrary Boolean function f.
- The positions with km tokens in the time-heap will correspond to the state of the CA at time m.

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- Let (a_1, a_2) denote a given position, where a_1 is the number of tokens in the tape-heap and a_2 is the number of tokens in the time-heap,
- Then the set of available moves is given by $\mathcal{M}_i$, where $i \equiv a_2 \pmod{k}$.

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- The set consisting of & and $\sim$ (negation) is a complete set of connectives in propositional logic.
- For instance the function $f(x, y) = x \oplus y$ can be expressed as
$$x \oplus y = [[xy][[x][y]]].$$

Computing the function f

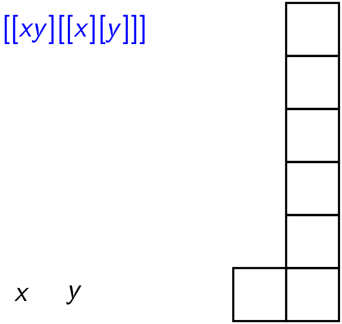

Figure: A modular game computing $f(x, y) = x \oplus y$ in five steps. The arrows indicate move options. The value of each cell is the bracket of the values of all its options.

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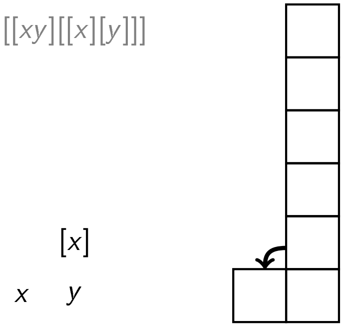

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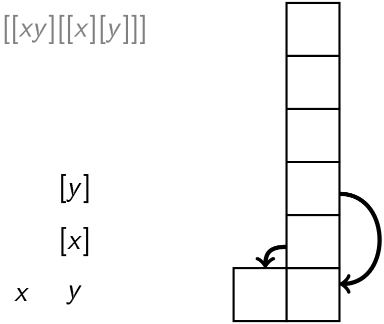

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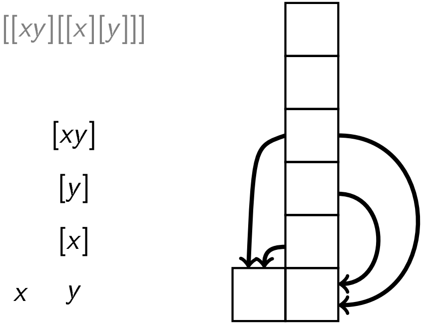

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$[y]$

$[x]$

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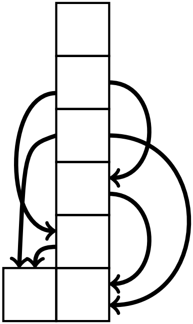

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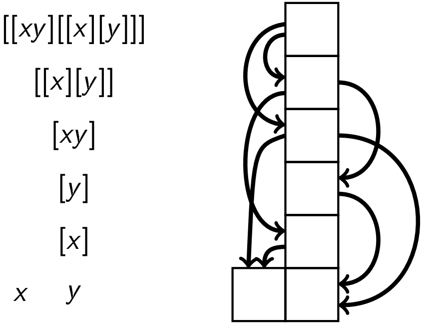

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- $\mathcal{M}_2 = \{(0, -2)\}$
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The construction in the last figure corresponds to the modular game

- $\mathcal{M}_3 = \{(0, -3), (-1, -3)\}$
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and its moves

The construction in the last figure corresponds to the modular game

- $\mathcal{M}_4 = \{(0, -2), (0, -3)\}$
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- $\mathcal{M}_1 = \{(-1, -1)\}$

and its moves

The construction in the last figure corresponds to the modular game

- $\mathcal{M}_0 = \{(0, -1), (0, -2)\}$
- $\mathcal{M}_4 = \{(0, -2), (0, -3)\}$
- $\mathcal{M}_3 = \{(0, -3), (-1, -3)\}$
- $\mathcal{M}_2 = \{(0, -2)\}$
- $\mathcal{M}_1 = \{(-1, -1)\}$

The P-positions

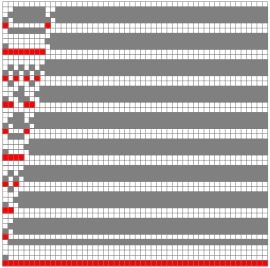

Figure: A modular game emulating $f(x,y) = x \oplus y$. Here the P-positions with fewer than 50 tokens in each heap are represented by filled squares. Rows corresponding to $a_2 \equiv 0 \pmod 5$ are highlighted by drawing the P-positions in red.

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- This need not be the case, but we want a single match in the gadget to follow the time-heap modulo k.
- The heaps of the gadget may be numbered $0, \ldots, k-1$

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- The tape- and time-heaps are affected as given by $\mathcal{M}_i$,
- A match is added to the heap of the gadget corresponding to the new modulus of the time-heap.

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- if and only if the patterns we ask for do not "trivially" occur for positions violating (i) or (ii).

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- Choose some large number N, and allow any move that transfers two tokens in the gadget to two other heaps and removes any number smaller than N from the two main heaps.
- Since the number of tokens in the gadget will never change, this will give a trivial periodic pattern of P-positions in the main heaps.

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The gadget contains exactly one match, but this match does not correspond to the number of tokens in the time-heap modulo k.

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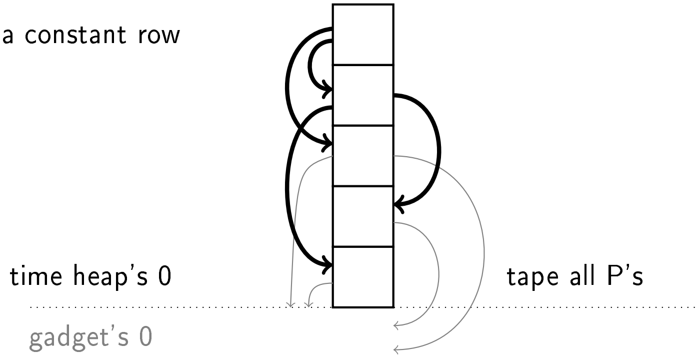

Figure: The gadget "thinks" the time heap is congruent to zero when it is not. The gray moves will not be possible.

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- If constant 1 (P-positions), then the following behavior will be the same as if the gadget was in phase, since the computation now starts from a row of $\cdots 000111 \cdots$

Conclusion

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- Hence it is undecidable whether a given finite pattern of P-position occurs in our invariant heap game.

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- We guess that three heaps suffice, and perhaps even two.
- Do we need to be able to add tokens to heaps in order to achieve undecidability?