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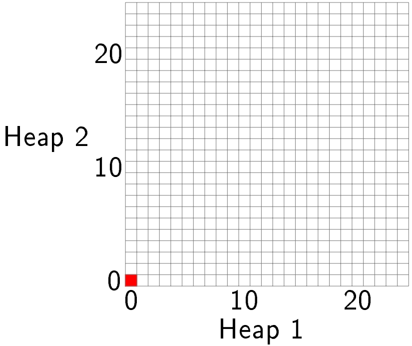

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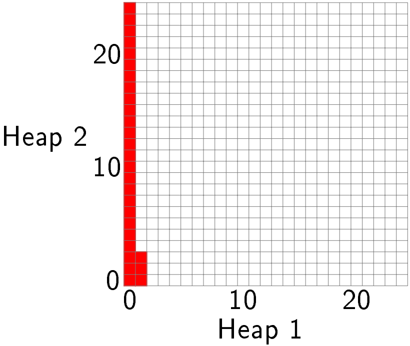

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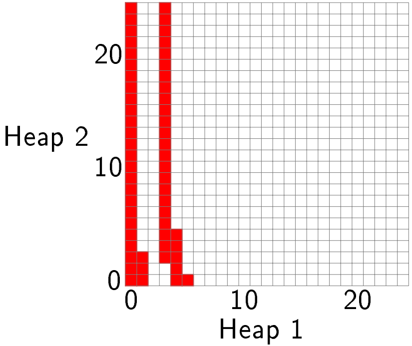

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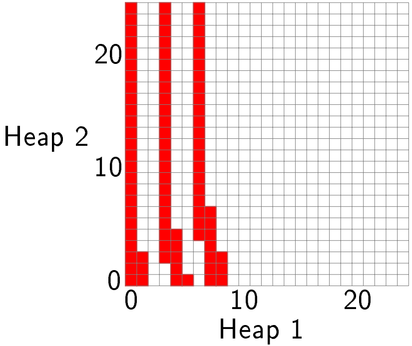

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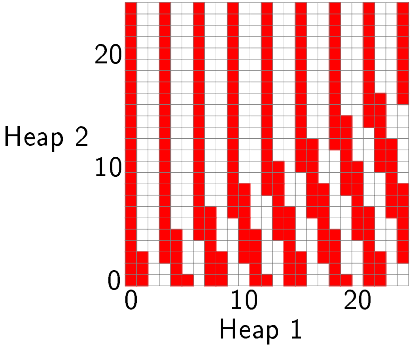

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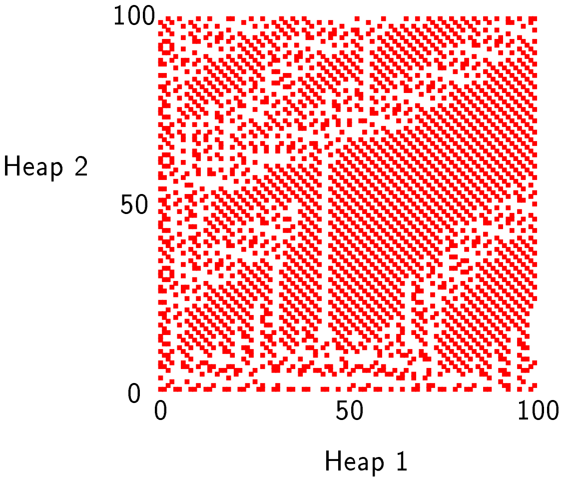
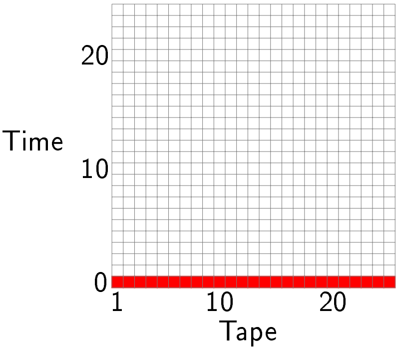


Figure: Initial $\mathcal{P}$-positions of the game given by $\mathcal{M} = \{(0,-2), (-2,0), (2,-3), (-3,2), (-5,4), (-5,-2), (-4,-3), (-1,-4)\}$.

3. The rule 60 CA



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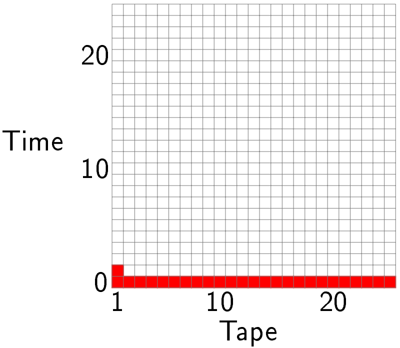

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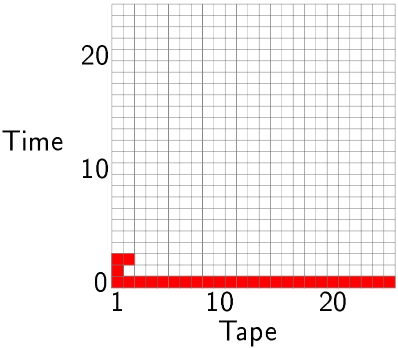

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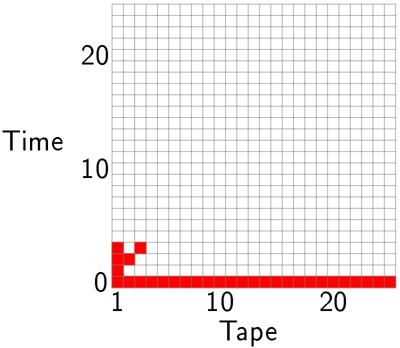

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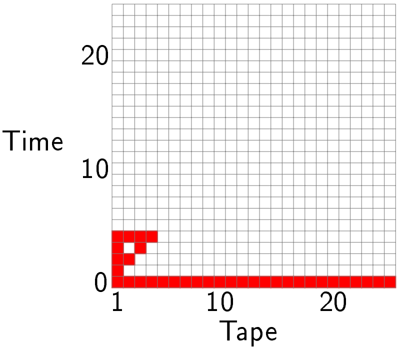

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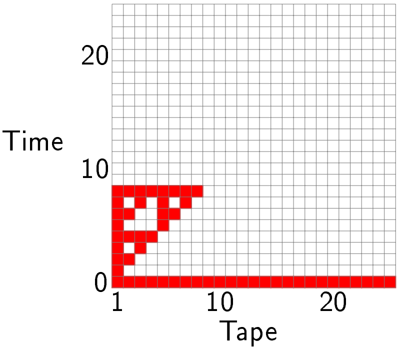

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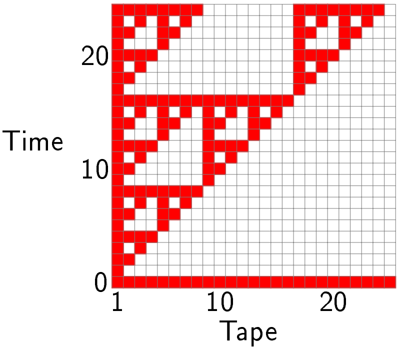

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- $f(x, y, z) = 0$ iff $x = y = z = 1$ or $x = y = 0$.

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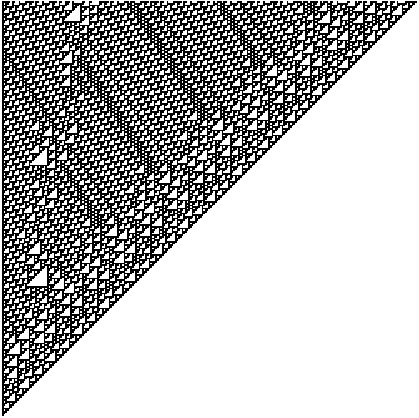

Figure: CA rule 110, time flows upwards, initial condition a single "1".

5. Computing the function f

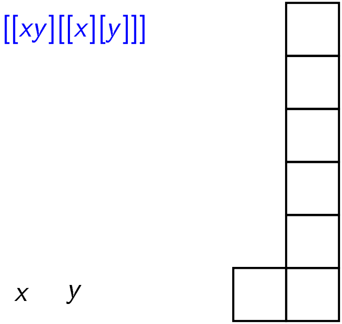

Figure: A modular game computing $f(x, y) = x \oplus y$ in five steps. The arrows indicate move options. The value of each cell is the bracket of the values of all its options.

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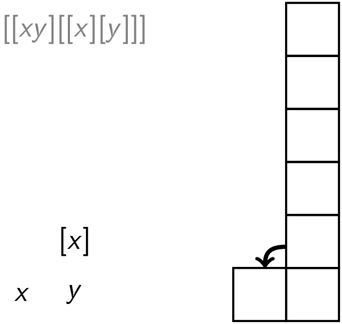

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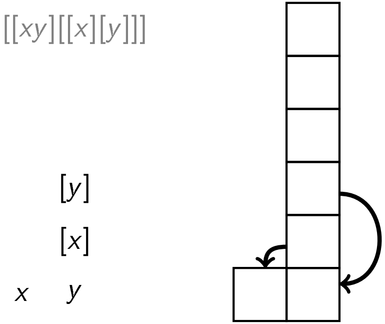

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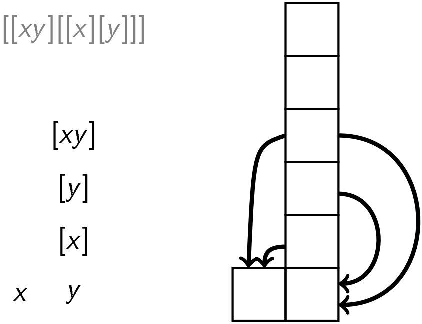

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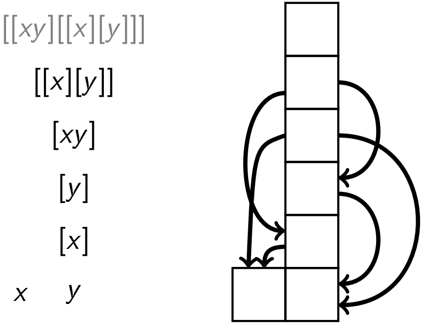

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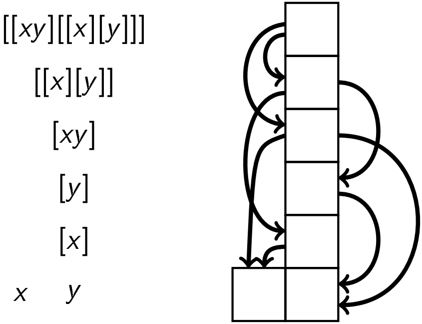

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6. via the move sets

The construction in the last figure corresponds to the modular game

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- $\mathcal{M}_4 = \{(0, -2), (0, -3)\}$
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6. via the move sets

The construction in the last figure corresponds to the modular game

- $\mathcal{M}_0 = \{(0, -1), (0, -2)\}$
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7. The $\mathcal{P}$-positions of this modular game

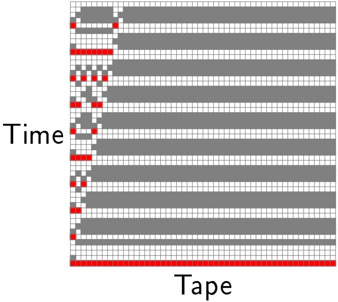

Figure: A modular game emulating $f(x, y) = x \oplus y$. Here the $\mathcal{P}$-positions with fewer than 50 tokens in each heap are represented by filled squares. Rows corresponding to $a_2 \equiv 0 \pmod 5$ are highlighted by drawing the $\mathcal{P}$-positions in red.

8. The check for "101"

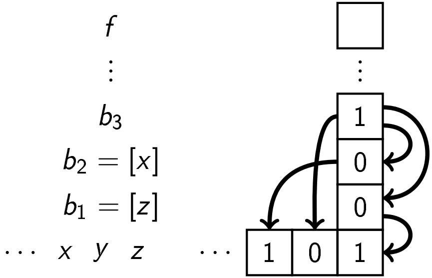

Figure: A modular game $\mathcal{G}'$ with input $\ldots xyz$ computing a function f in a few steps. The first two boxes b_1 and b_2 invert x and z. The third box b_3 is a 1 (=$\mathcal{P}$-position) if and only if $xyz = 101$.

9. $\mathcal{P}$-positions differ iff CA contains "101"

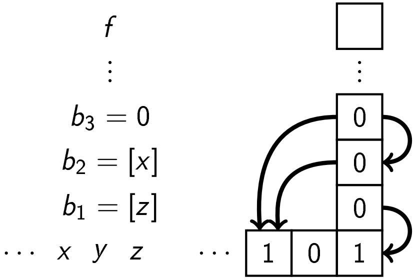

Figure: A modular game $\mathcal{G}''$ whose output function f is identical to that of $\mathcal{G}'$. The only difference is that $\mathcal{G}''$ does not check for the pattern 101. The box b_3 contains a 0 independently of the input xyz to f.

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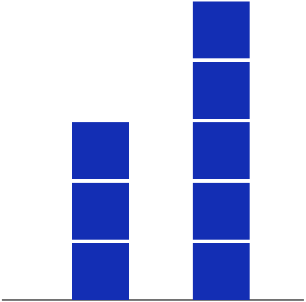

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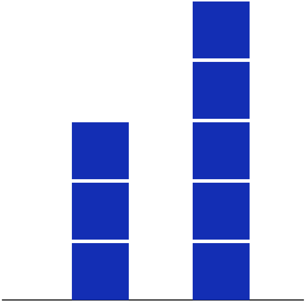

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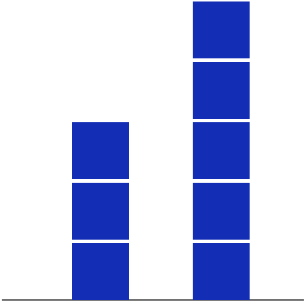
$\mathcal{P} = (0,0),(1,1),(2,2),\ldots$



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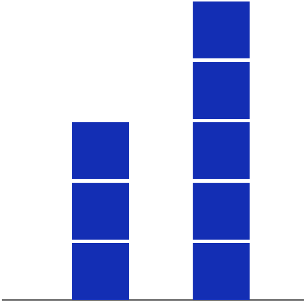

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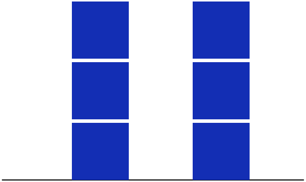

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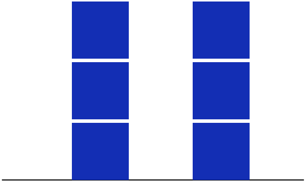

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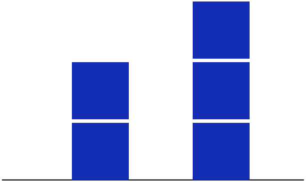

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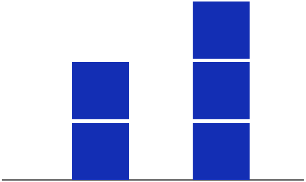

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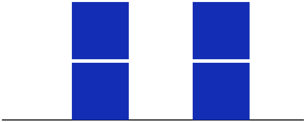

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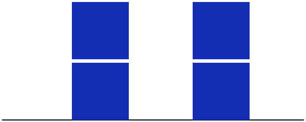

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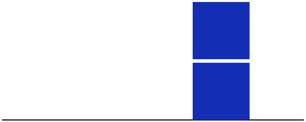

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12. Move-size dynamic rules: Imitation Nim (Larsson 2009)

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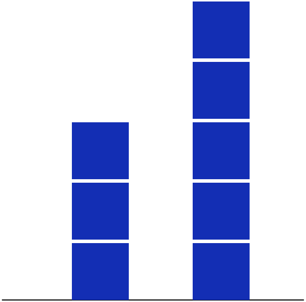

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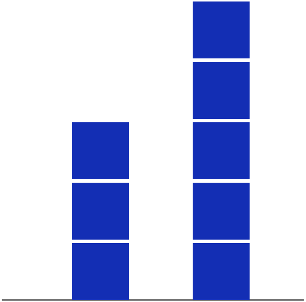

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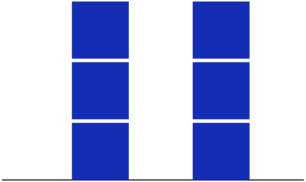

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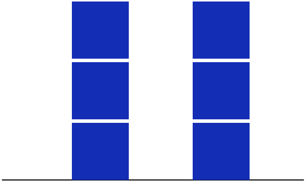

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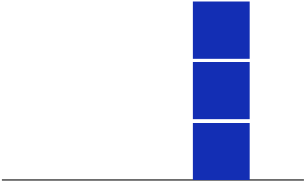

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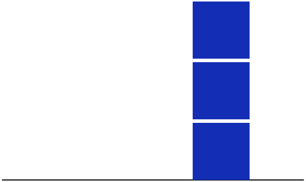

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- Then the second player wins by removing all tokens from one heap.
- In fact, the $\mathcal{P}$-positions are as in Wythoff's Nim (1907).

13. The triangles in the rule 110 CA...

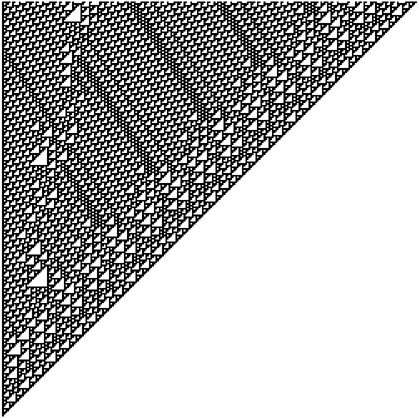


Figure: CA rule 110, time flows upwards, initial condition a single “1”.

13. ...have the same shapes as those in the rule 60 CA

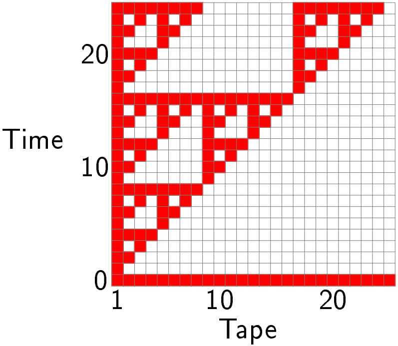
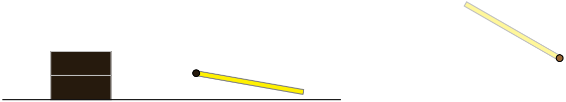


Figure: The CA given by $f(x, y) = x \oplus y$ (Wolfram's rule 60).

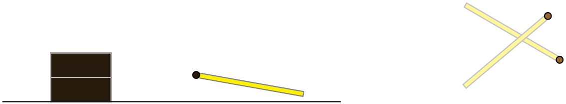
14. A terminal position of the rule 60 game

The previous player removed the rightmost match. Hence $m_p = 1$ and so $0 \leq t \leq 1$. The next player cannot remove both tokens; neither the final match.



15. A next player winning position

The previous player removed the rightmost two matches. Hence $0 \leq t \leq 2$. Both tokens can be removed and the final match.

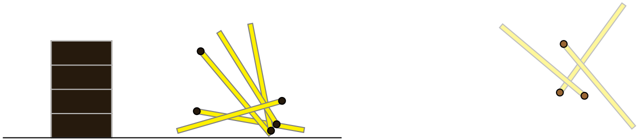


15. A next player winning move

The previous player removed the rightmost two matches. Hence $0 \leq t \leq 2$. Both tokens can be removed and the final match.

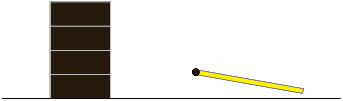
16. The position $(4, 5, 3)$

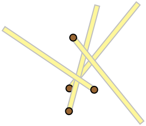
At most 3 tokens can be removed from this position. Suppose that Player 1 removes all matches but one. Then player 2 removes all tokens together with the final match.



16. The position $(4, 5, 3)$

At most 3 tokens can be removed from this position. Suppose that Player 1 removes all matches but one. Then player 2 removes all tokens together with the final match.





16. The position $(4, 5, 3)$

At most 3 tokens can be removed from this position. Suppose that Player 1 removes all matches but one. Then player 2 removes all tokens together with the final match. In fact...

17. The rule 60 game position $(4, 5, 3)$ is in $\mathcal{P}$

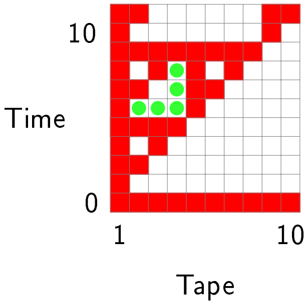


Figure: The green pattern indicates a second player winning position $\#tape = 4$, $\#time = 5$, $m_p = 3$.

17. so is (4, 5, 2)

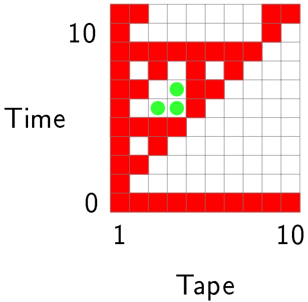


Figure: The green pattern indicates a second player winning position $\#tape = 4$, $\#time = 5$, $m_p = 2$.

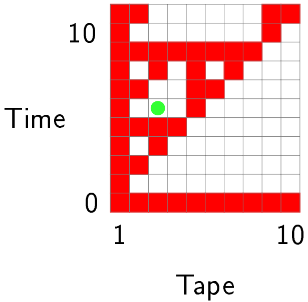


Figure: The green pattern indicates a second player winning position $\#tape = 4$, $\#time = 5$, $m_p = 1$.

17. but not $(4, 6, 1)$

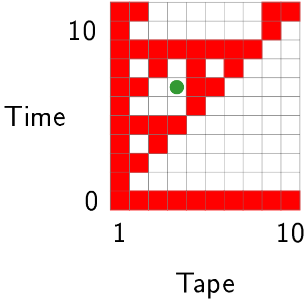


Figure: A first player winning position $\#tape = 4$, $\#time = 6$, $m_p = 1$ (remove one match or one match and one token).

17. neither is $(3, 5, 4)$

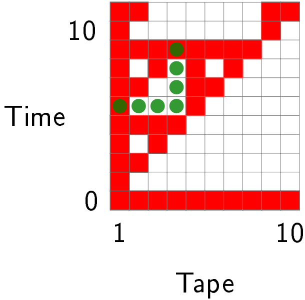


Figure: A first player winning position $\#tape = 3$, $\#time = 5$, $m_p = 4$ (remove all matches and tokens)

17. nor (4, 4, 2)

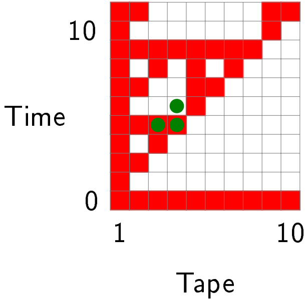


Figure: A first player winning position $\#tape = 4$, $\#time = 4$, $m_p = 2$ (remove all but one match and no token).

18. The rule 110 game

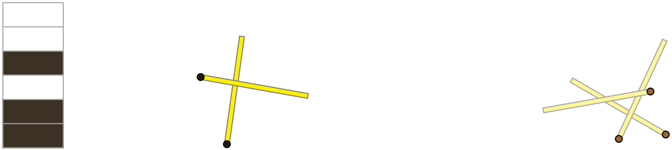


Figure: The position ("110100", 2, 3).

A position of the rule 110 game. The second player wins, see green circle:

18. The rule 110 CA and some game positions

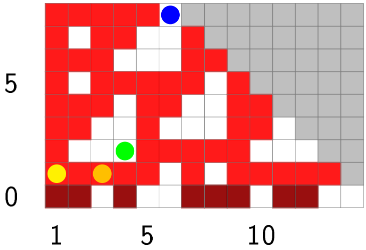


Figure: Some rule 110 game positions (m_p omitted) for the finite ending condition "11010011101100" together with CA updates. Example: green position ("110100", $2, m_p$) second player wins if $1 \leq m_p \leq 3$. For the other positions the first player wins independent of m_p.

19. Undecidable F glider occurrence in $\underline{L}C\underline{R}$ rule 110 CA

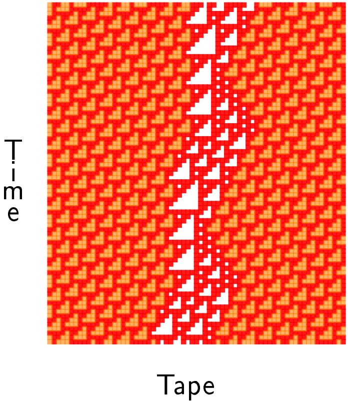


Figure: An F glider embedded in Rule 110 ether.

20. An undecidable path of moves

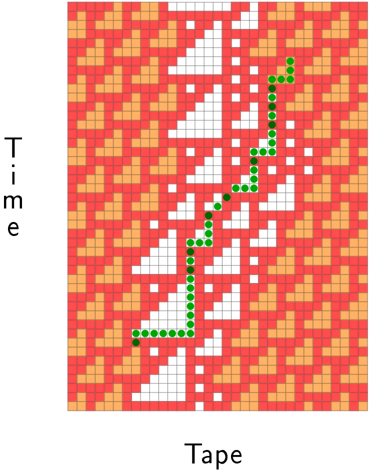


Figure: An optimal sequence of consecutive moves in the rule 110 game traversing an F glider.