Sequences and games generalizing the combinatorial game of Wythoff Nim

... and the game of Imitation Nim

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Table of contents

Three papers

I: Permutations of the natural numbers with prescribed difference multisets (published in Integers, Volume 6 (2006), article A3),

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- II: 2-pile Nim with a restricted number of move-size dynamic imitations (accepted for publication in Integers, Volume 9 (2009), article G4),
- III: Restrictions of m-Wythoff Nim and p-complementary Beatty sequences (accepted for publication in Games of no Chance 2008).

Combinatorial Number Theory

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- Here: A pair of sequences of non-negative integers (Paper II, III).

Which properties are satisfied by the P-set, $\mathcal{P}$?

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(Uii) $x' = (x_1, x_2, \ldots)$ and $y' = (y_1, y_2, \ldots)$ are complementary on $\mathbf{N}$, that is, $x' \cup y' = \mathbf{N}$ and $x' \cap y' = \emptyset$.

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Notice that (Uii) and (Uiii) implies $x_0 = y_0 = 0$.

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Theorem (Uniqueness theorem)

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(Conway 1976) Let $X \subset \mathbf{N}_0$. Then mex$(X) = \min(\mathbf{N}_0 \setminus X)$, the least non-negative integer not in X.

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Then, by definition, a and b are increasing, complementary and (Uiii) is trivially satisfied. This algorithm has exponential complexity in $\log(n)$. (A. Fraenkel, many papers, first 1973?)

The first few (a_n, b_n) and (b_n, a_n)

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Rayleigh's theorem

Let $\mathbf{R}$ denote the real numbers.

Definition

Let α be a positive irrational, $\gamma \in \mathbf{R}$. Then $(\lfloor \alpha n + \gamma \rfloor)_{n \in \mathbf{N}_0}$ is a Beatty sequence.

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Proposition (Rayleigh's theorem)

(Rayleigh 1894, Beatty 1926, Ostrowski & Hyslop 1927) Suppose that $(\lfloor \alpha n \rfloor)_{n \in \mathbf{N}}$ *and* $(\lfloor \beta n \rfloor)_{n \in \mathbf{N}}$ *are Beatty sequences. Then they are complementary if and only if* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

'Ostrowski/Hyslop proof'

Proof. Let us, for each $n \in \mathbf{N}$, estimate the total number of elements, say X, in the two sequences $\leq n$. Since α and β are irrational, we have the following bounds,

$$\left\lfloor \frac{n}{\alpha} \right\rfloor + \left\lfloor \frac{n}{\beta} \right\rfloor < X < \left\lceil \frac{n}{\alpha} \right\rceil + \left\lceil \frac{n}{\beta} \right\rceil .$$

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The Wythoff pairs

Let $\Phi = \frac{1+\sqrt{5}}{2}$ denote the golden ratio. Then $\frac{1}{\Phi} + \frac{1}{\Phi+1} = 1$ and Φ is irrational so

$$A := (\lfloor \Phi n \rfloor)_{n \in \mathbf{N}}$$

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By blue A and B satisfy (U). The answer of the following question may be determined in polynomial time in $\log(n)$. Given a pair $(X, Y), X \leq Y$, is there any n such that $(A_n, B_n) = (X, Y)$?

Recall the set $\mathcal{P}$

If the Uniqueness theorem holds, clearly $a = A$ and $b = B$. But this can also be proved by elementary means, by verifying that, for all n, $a_n = A_n$. This is done for a more general case in Lemma 3.3, Paper III.

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- Who wins? (most points/last player to move).

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Perfect strategy—Game complexity.

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- Partizan. The options depend on whose turn it is. (Chess, Go)
- Impartial. The options are the same for both players (Nim, Wythoff Nim, Octal games (such as Kayles), Children games: Subtraction games (such as "21"), Geography)

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Normal play

The winning condition is given by:

The last player to move wins. This is called normal play.

Terminology

Let G be an impartial game. Then G is P if the previous player, the player who is not in turn to move, wins and N if the next player wins. The set of all P-positions of G is denoted by $\mathcal{P} = \mathcal{P}(G)$ and ditto for $\mathcal{N}$. Denote the set of options of G by $F(G)$, that is, there is a move $G \to X$ if and only if $X \in F(G)$.

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Let G be an impartial game. Then G is P iff $F(G) \subset \mathcal{N}$ and G is N iff $F(G) \cap \mathcal{P} \neq \emptyset$.

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It follows that each terminal position is P. All our games can be interpreted as so-called take away games: Given rules of game, the players alternate in removing tokens from a finite number of heaps.

The Impartial game of k-pile Nim

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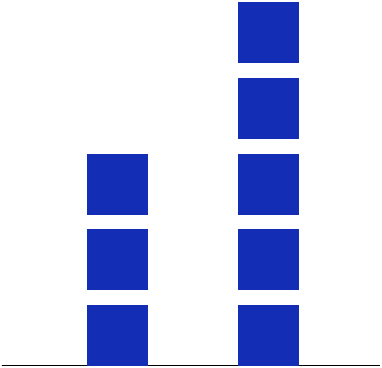
Theorem (C. Bouton (1902))

Let x_i, $i \in \{0, 1, \ldots, k\}$ denote the respective pile-heights of k-pile Nim. Then $(x_1, x_2, \ldots, x_k)$ is a P-position if and only if $\oplus_2 x_i = 0$.

Example: 2-pile Nim



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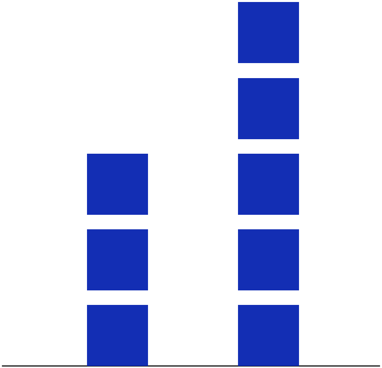

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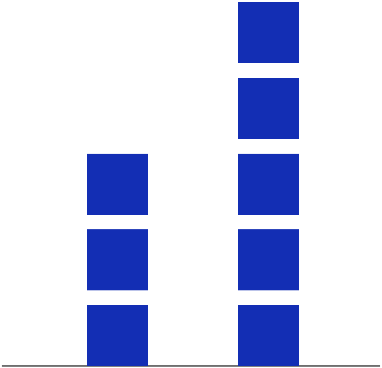

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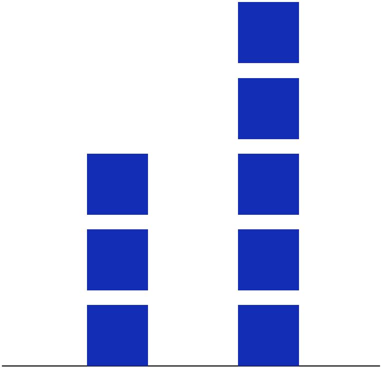

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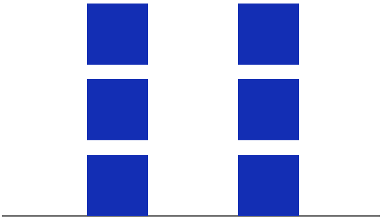

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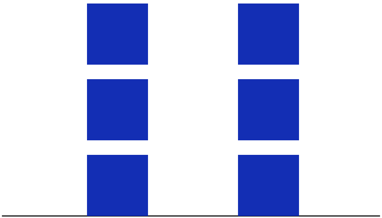

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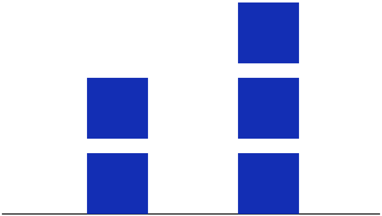

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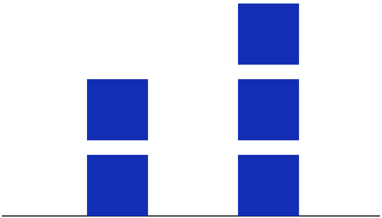

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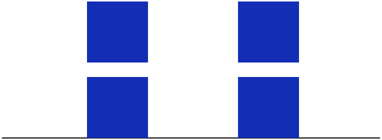

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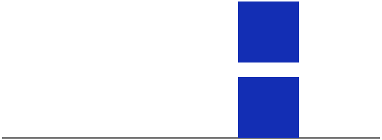

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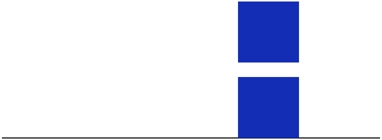

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We arrive at a new game, a restriction of 2-pile Nim, with a new winning strategy. What is this strategy?

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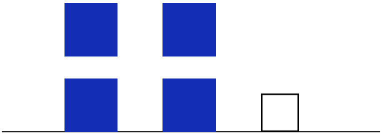

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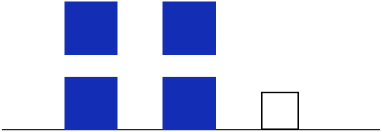

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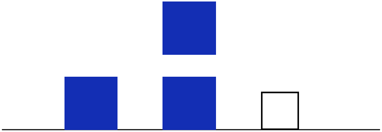

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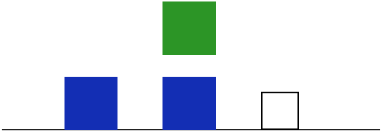

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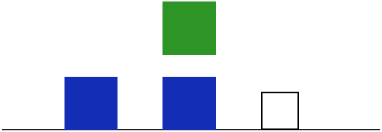

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- So $(2,2)$ is N. Unlike in 2-pile Nim.

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Proof. (sketch) Denote with $\mathcal{P}' = \{\{a_n, b_n\}\}$ and with $\mathcal{N}' = \mathbf{N}_0 \times \mathbf{N}_0 \setminus \mathcal{P}'$.

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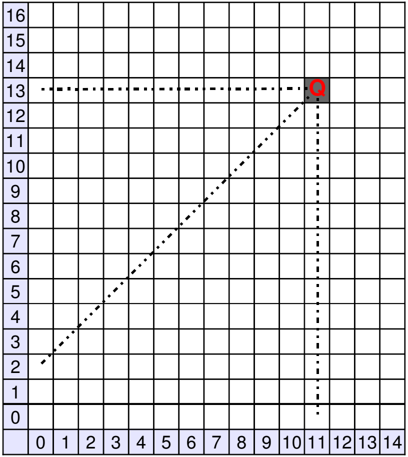
The game is maybe more known as the impartial game "Corner the Queen" (Rufus P. Isaacs, 1960), where the two players alternate in moving one single Queen on a (large) Chess-board—aiming to be the player who puts it in the lower left corner. The distance to this corner must by each move decrease (L^1 norm).

Moves of Wythoff Nim



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Where are the new P-positions?

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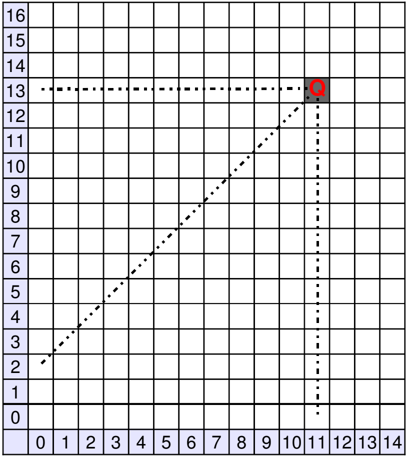

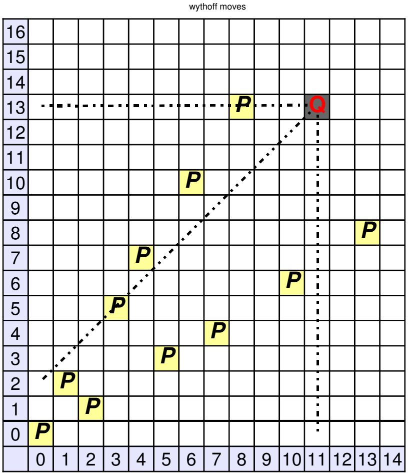
wythoff moves
16
15
14
13
12
11
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9
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7
6
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4
3
2
1
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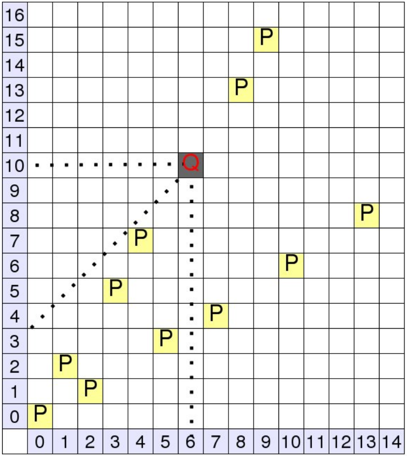
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
P
Q


16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
P
Q

The solution of Wythoff Nim

Theorem (W.A.Wythoff (1907))

$\mathcal{P}(WN) = \{\{A_i, B_i\} \mid i \in \mathbf{N}_0\}$.

A game with memory

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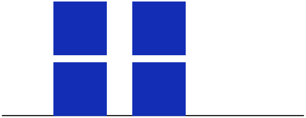
Fix a $p \in \mathbf{N}$. We may relax the imitation rule: Allow at most $p-1$ consecutive imitations by one and the same player. Prohibit the p:th imitation. This game we call $(p,1)$-Imitation Nim. Notice that going from Imitation Nim to $(p,1)$-Imitation Nim gives more next player options.

A Diagonal Blocking Wythoff Nim

The “dual of Imitation Nim was defined already in Paper I, The previous player may 'block off' at most one option from the next player's 'diagonal set' of options.

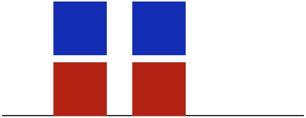
Example: One diagonal option may be blocked

- The starting position is (2,2).



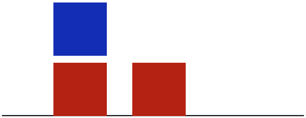
Example: One diagonal option may be blocked

- The starting position is (2,2).
- Bob chooses to block off (0,0);



Example: One diagonal option may be blocked

- The starting position is (2,2).
- Bob chooses to block off (0,0);
- Alice moves to (1,1) and makes the obvious block;



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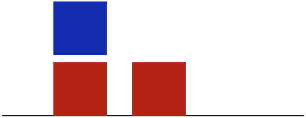

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- ...just like in (2,1)-IN.

Thank you!