Invariant and dual games resolving the Duchêne-Rigo conjecture

Urban Larsson, joint with P. Hegarty and A. S. Fraenkel,
A discrete seminar at Chalmers, University of Gothenburg.

October 5, 2010

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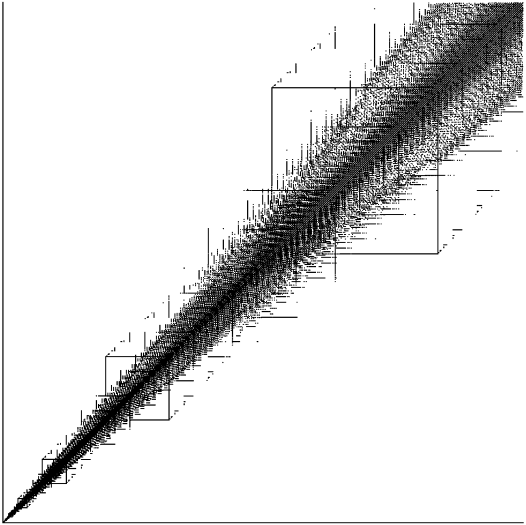

The Duchêne-Rigo conjecture on invariant games

The Rayleigh/Beatty theorem (1894/1927)

We say that the ordered pair (α, β) is a Beatty pair if $\alpha < \beta$ are positive irrationals with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Observation, then $1 < \alpha < 2 < \beta$.

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Conjecture (Duchêne-Rigo, 2009)

Suppose that (α, β) is a Beatty pair. Then there exists an invariant take-away game with its set of P-positions identical to $\{(\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor), (\lfloor n\beta \rfloor, \lfloor n\alpha \rfloor) \mid n \in \mathbb{N}_0\}$.

What is a take-away game?

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- Nim-sum (binary addition without carry) = 0.

Playing 2-pile Nim

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- Starting position, $(3, 5)$.

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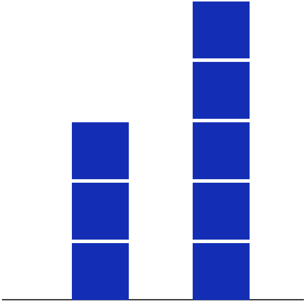

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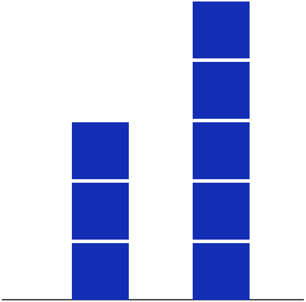

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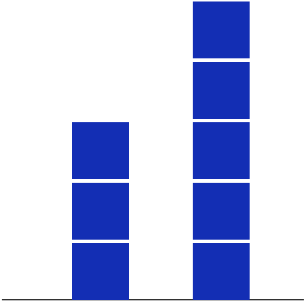

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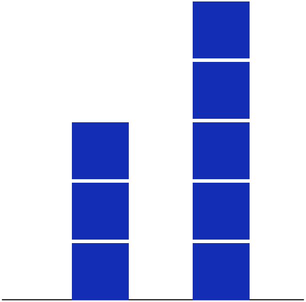

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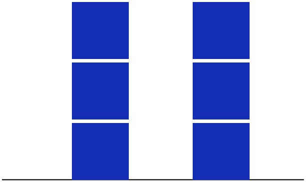

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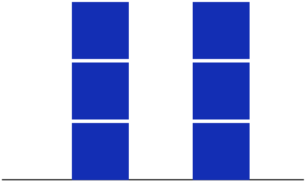

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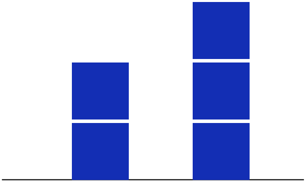

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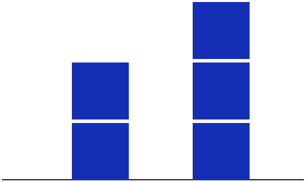

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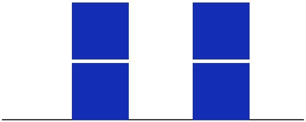

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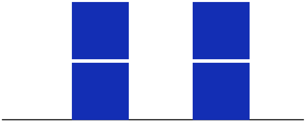

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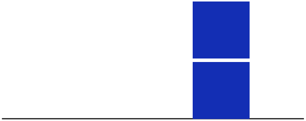

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- Nim!

Nim, as an invariant subtracion game

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- Pile-heights decreases, a subtraction game.
- Invariance of moves.

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Let $\mathcal{B} = \mathbb{N}_0 \times \mathbb{N}_0$. A 2-pile subtraction game G is said to be invariant if,

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is a legal move whenever

$$(\boldsymbol{y} \oplus \boldsymbol{r}) \to \boldsymbol{y}$$

is.

The invariant moves

For invariant games we refer to the set

$$\mathcal{M}(G) := \{\boldsymbol{r} \in \mathcal{B} \mid \boldsymbol{r} \to \boldsymbol{0} \text{ is a legal move}\}$$

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as the set of all (invariant) moves.

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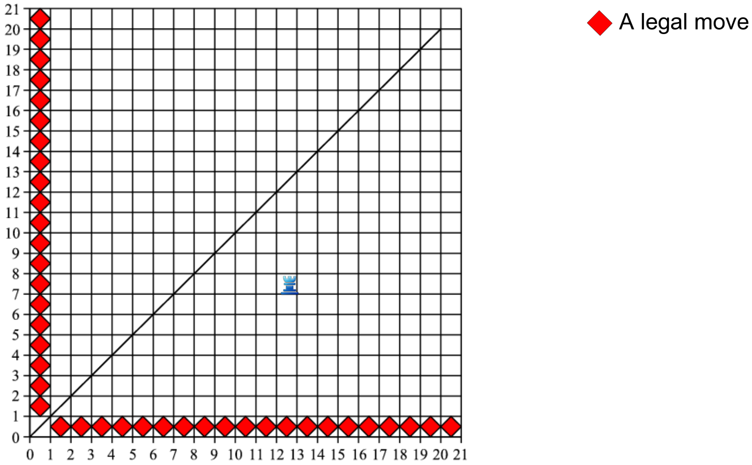

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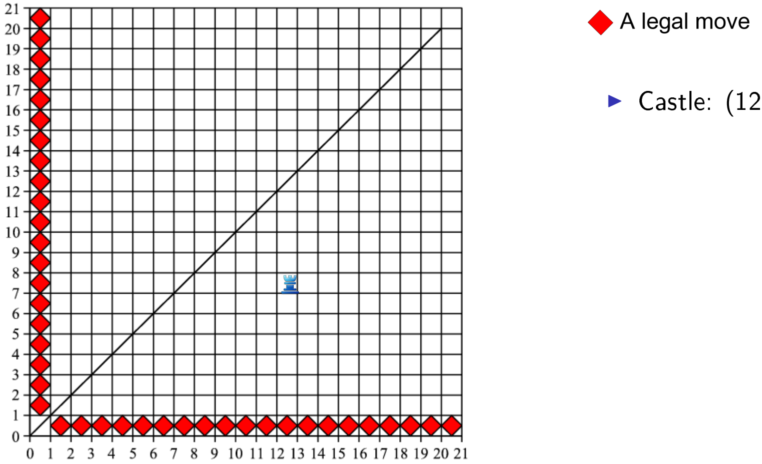
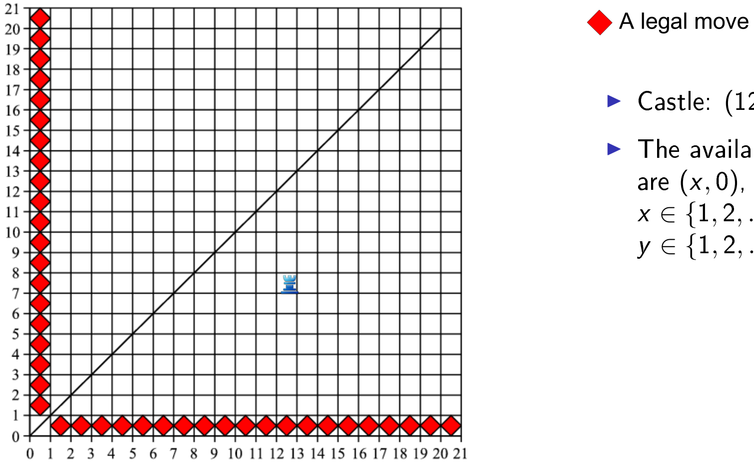


- Castle: $(12, 7)$,

Nim, as an invariant subtraction game



- Castle: $(12, 7)$,
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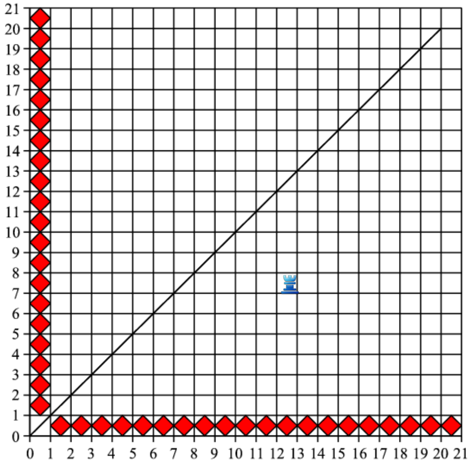

A legal move

- Castle: $(12, 7)$,
- The available moves are $(x, 0)$, $(0, y)$ $x \in \{1, 2, \ldots, 12\}$, $y \in \{1, 2, \ldots, 7\}$
- A Nim option: $(12, 7) \ominus (x, 0) = (12 - x, 7)$ or $(12, 7) \ominus (0, y) = (12, 7 - y)$

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- The first player to move wins if and only if the game is N.
- $\mathcal{P}(G)$ (resp. $\mathcal{N}(G)$) is the collection of P- (resp. N-) positions of G.

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- which, by the definition of P, is impossible.

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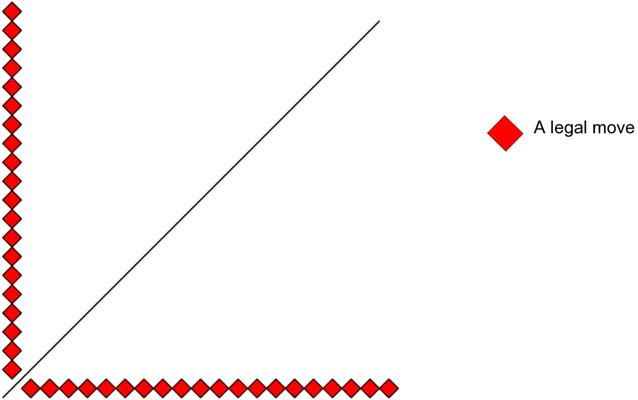
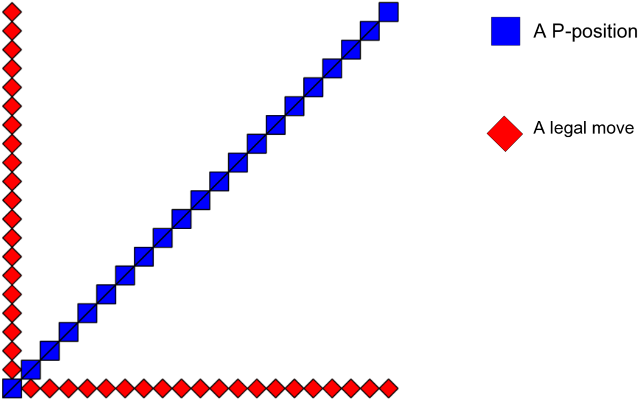
If G is a (not necessarily invariant) game, then we can define an invariant game $G^\star$ on the same game board by setting

$$\mathcal{M}(G^\star) := \mathcal{P}(G) \setminus \{\mathbf{0}\}.$$

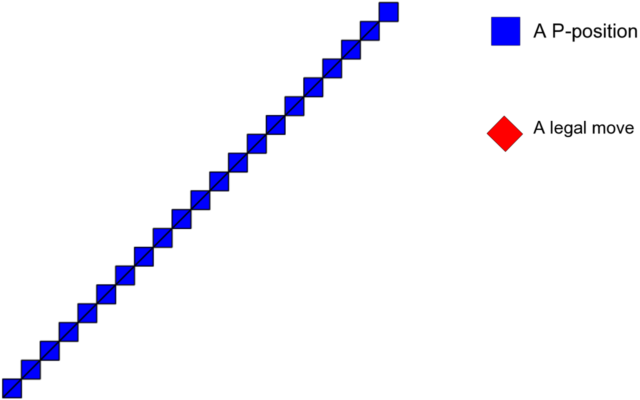
Example: Nim



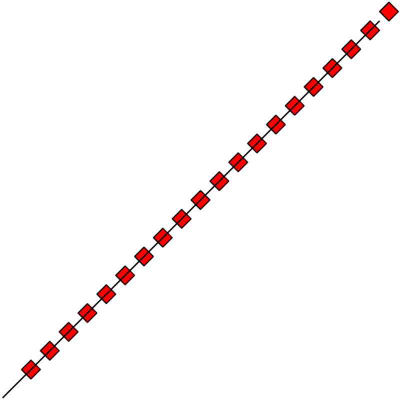
Nim and its P-positions

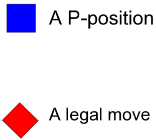


The P-positions of Nim

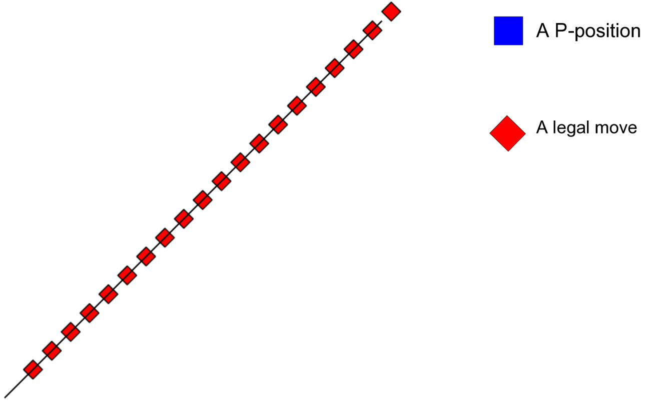


The moves of Nim ⋆

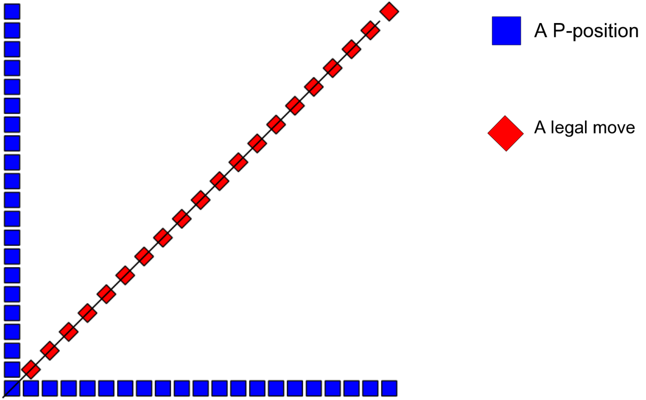




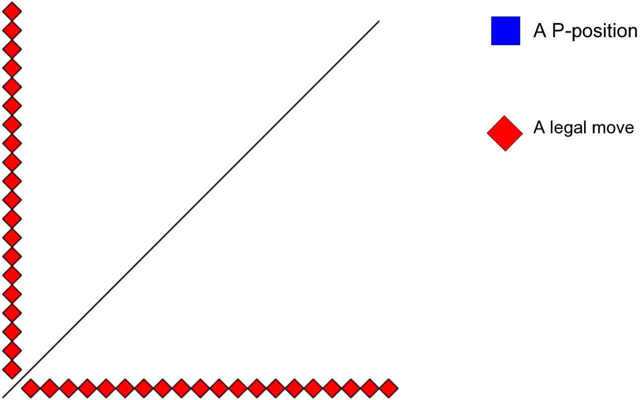
Wait! What are the P-positions of Nim ⋆?



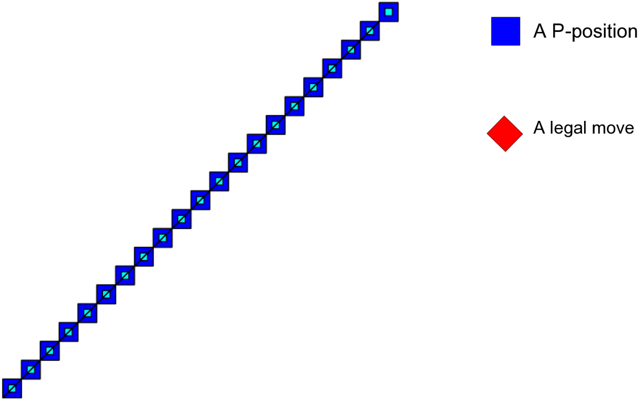
The moves and P-positions of Nim $\star$



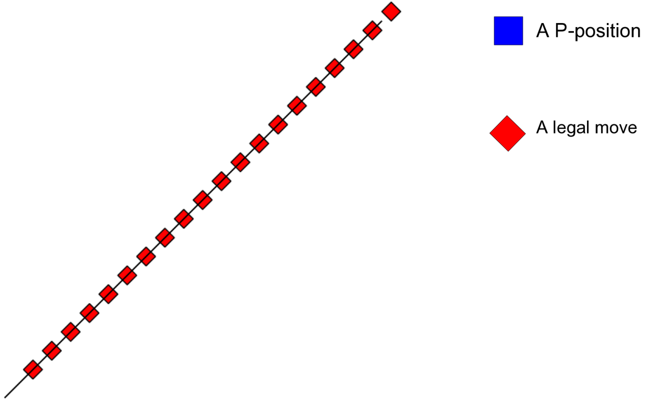
The game of (Nim ⋆) ⋆ equals Nim



Hence, the P-positions are identical



Thus, Nim ⋆ may be regarded as the 'dual' of Nim



Complementary Beatty sequences

A pair of sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of positive integers is said to be complementary if $\{x_n\} \cup \{y_n\} = \mathbb{N}$ and $\{x_n\} \cap \{y_n\} = \emptyset$.

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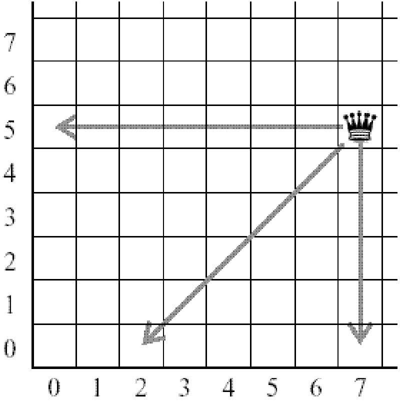
Wythoff Nim

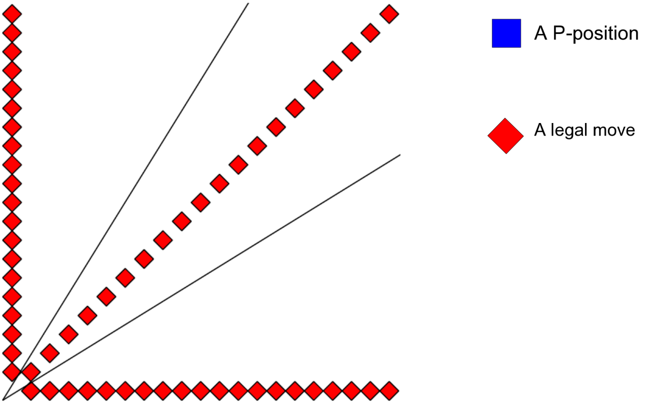
A golden instance of Beatty pairs

Let $\phi := \frac{1+\sqrt{5}}{2}$ denote the Golden ratio. Our next example concerns the Beatty pair (ϕ, ϕ^2).

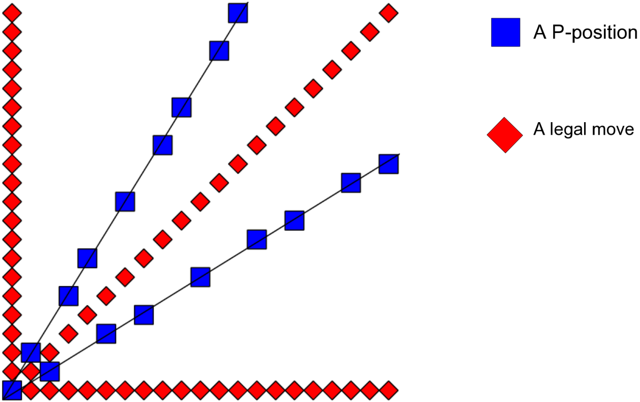
Wythoff Nim (1907), 'Corner the Queen'



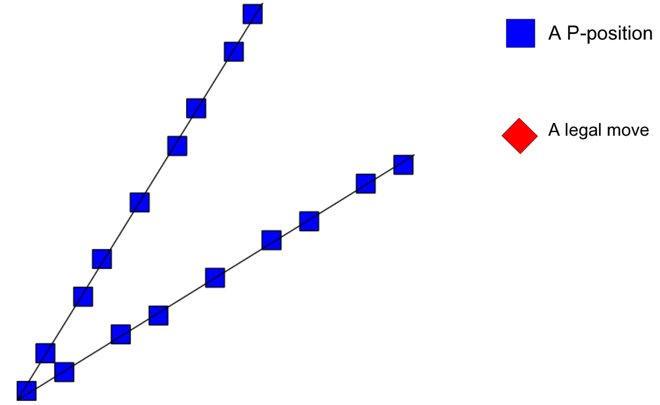
Wythoff Nim and the lines ϕx and $\frac{x}{\phi}$.



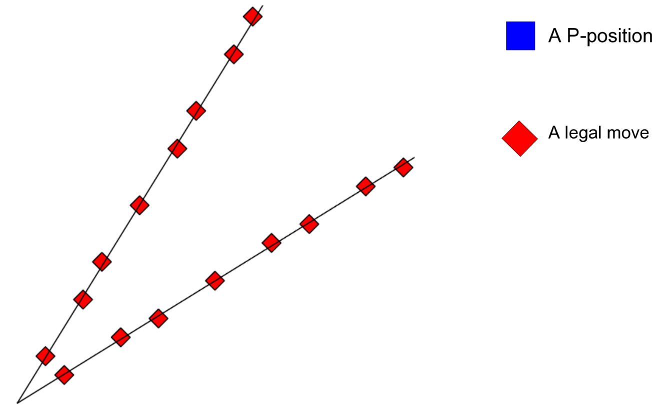
The moves and P-positions of Wythoff Nim



The P-positions of Wythoff Nim, $(\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor), n \in \mathbb{N}_0$



The initial moves of (Wythoff Nim)*



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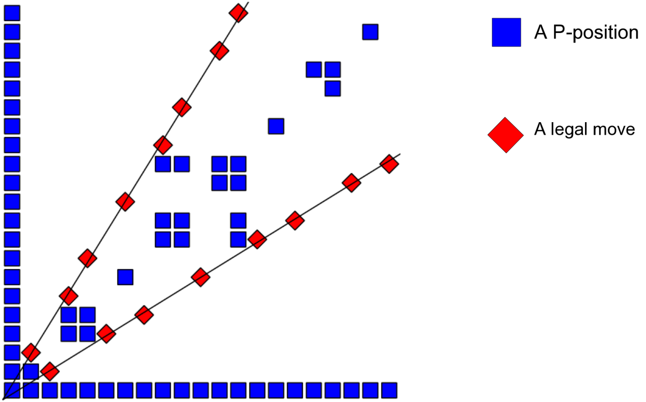

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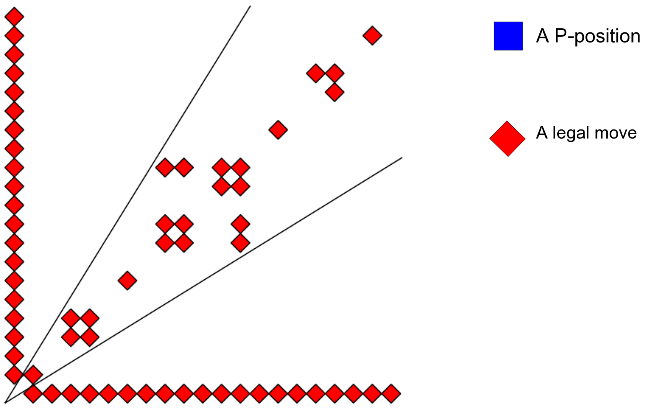

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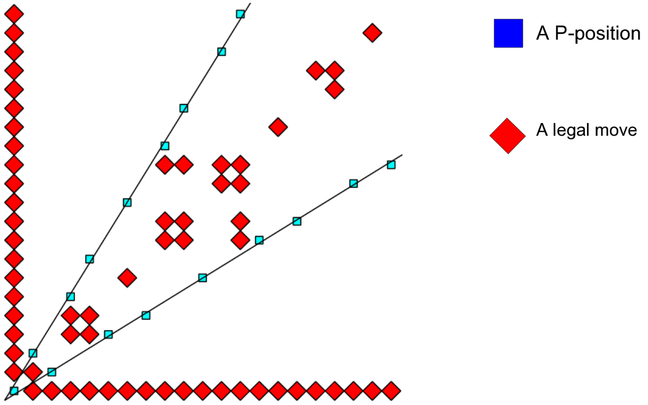
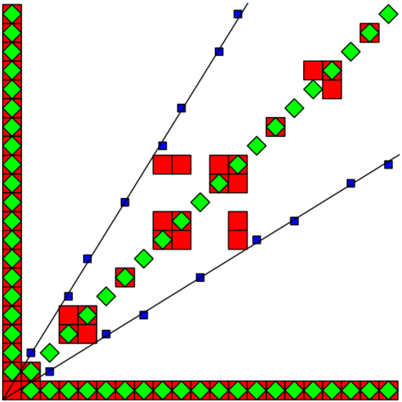
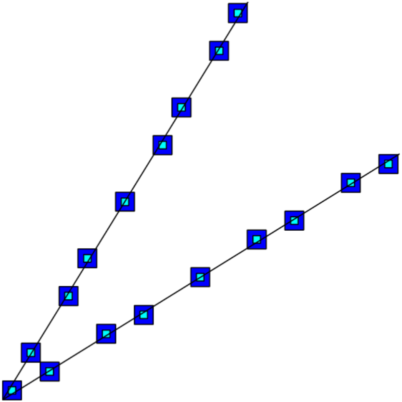


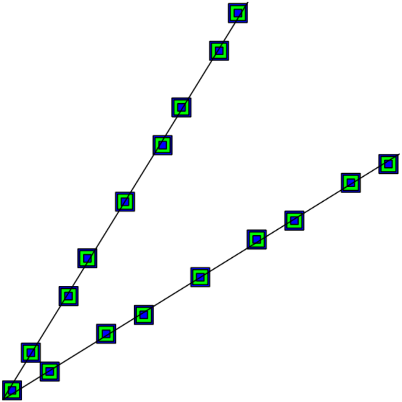
Wythoff Nim $\neq$ ((Wythoff Nim)*)*



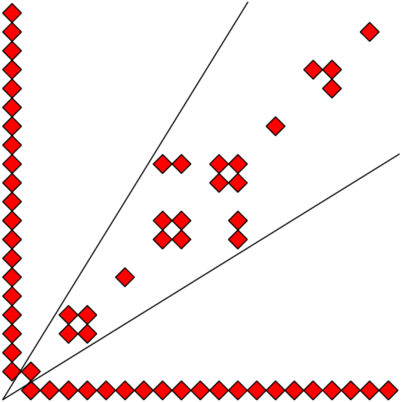
$\mathcal{P}(\text{Wythoff Nim}) = \mathcal{P}(((\text{Wythoff Nim})^\star)^\star)$

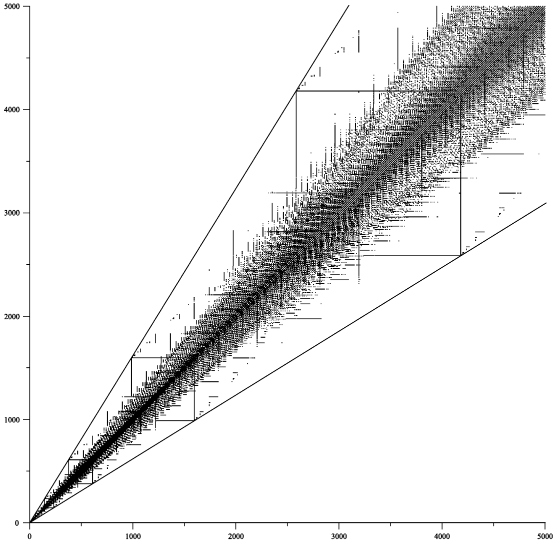


Identical P-positions of (Wythoff Nim)^{2k*}, $k \in \mathbb{N}_0$



Identical moves of (Wythoff Nim)^{2k*}, $k \in \mathbb{N}$. What are they? Is there a 'closed formula'?




5000
4000
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t-superadditive-complementarity, t-SAC

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Let $t \in \mathbb{N}$. We say that a sequence $(X_n)_{n\in\mathbb{N}_0}$ of non-negative integers is *t-superadditive* if, for all $m, n \in \mathbb{N}_0$,

$$X_m + X_n \leq X_{m+n}$$

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t-superadditive-complementarity, t-SAC

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The Main theorem

Suppose $\{(a_n, b_n) \mid n \in \mathbb{N}_0\}$ is b_1-SAC.

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and hence

$$(G^\star)^\star = G.$$

A consequence

Observation:

Any homogeneous Beatty sequence is 2-superadditive. Hence, if a and b is a pair of complementary homogeneous Beatty sequences, then the set $\{(a_n, b_n) \mid n \in \mathbb{N}_0\}$ is 2-SAC, hence b_1-SAC.

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And so..

The Duchêne-Rigo conjecture holds

Corollary

Let (α, β) be a Beatty pair. Then there exists an invariant game G such that $\mathcal{P}(G) = \{(\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor), (\lfloor n\beta \rfloor, \lfloor n\alpha \rfloor) \mid n \in \mathbb{N}_0\}$.

The structure of the proof of the Main theorem.

We need to prove that:

"$P \to N$":
For all $n \in \mathbb{N}$, if $(a_n, b_n) \to (s, t)$ is a legal move in $G^\star$, then (s, t) is not of the form (a_i, b_i) or (b_i, a_i), $i \in \mathbb{N}_0$.

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If $0 \leq s \leq t$ are integers and (s, t) is not of the form $(a_i, b_i), i \in \mathbb{N}_0$, then, for some $n \in \mathbb{N}_0$, there is a move of the form $(s, t) \to (a_n, b_n)$ or $(s, t) \to (b_n, a_n)$.

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The direction "$P \to N$".

For the other direction, a detailed analysis of b_1-SAC is required. However our first lemma is equally simple as elegant.

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Lemma (A P-position is never an invariant move)

Let G be an invariant Subtraction game. Then $\mathcal{M}(G) \cap \mathcal{P}(G) = \emptyset$.

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Goto non-b_1-SAC examples.

Facts of b_1-SAC sequences

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Suppose that $\{(a_n, b_n) \mid n \in \mathbb{N}_0\}$ is b_1-SAC. Then, for all $n \in \mathbb{N}_0$,

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- (b) *$k = b_n$ and $l \geq a_n$, or*
- (c) *$k = a_n$, $a_n = a_{n-1} + 1$ and $b_{n-1} \leq l < b_{n-1} + b_1$.*

A variant subtraction game

The Mouse game

A. Fraenkel recently studied a variant game, 'the Mouse game', whose P-positions are defined by complementary, so-called, inhomogeneous Betty sequences with rational moduli.

A variant subtraction game

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A. Fraenkel recently studied a variant game, 'the Mouse game', whose P-positions are defined by complementary, so-called, inhomogeneous Betty sequences with rational moduli. They are $(0,0)$ together with all positions of the form $(\lfloor\frac{3n}{2}\rfloor, 3n-1)$ and $(3n-1, \lfloor\frac{3n}{2}\rfloor)$, $n\in\mathbb{N}$.

The moves of the Mouse game

Move as in Wythoff Nim. In addition, if $y - x \equiv 0 \pmod 3$, then a player may move

$$(x, y) \to (w, z),$$

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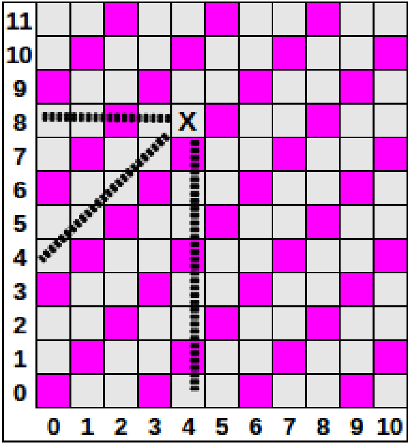
where $x - w > 0$, $y - z > 0$ and

$$\mid (x - w) - (y - z) \mid \leq 1.$$

The moves of the Mouse game



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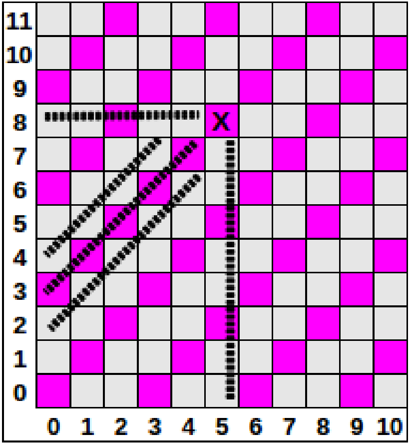
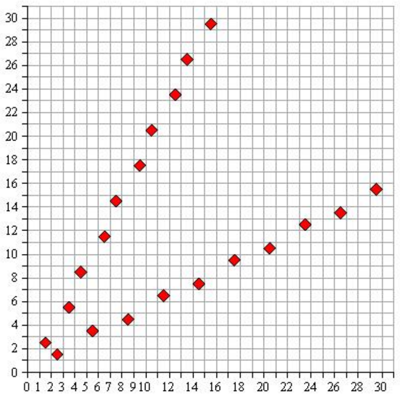


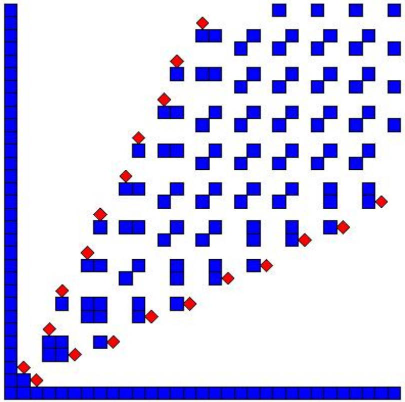
The dual of (Mouse game)*

It is not hard to prove that the set $\{(\lfloor \frac{3n}{2} \rfloor, 3n-1) \mid n \in \mathbb{N}\}$ is $b_1 - SAC$. Hence, by definition of $\star$ and our main theorem, we may define an invariant game, the 'Mouse trap', with identical P-positions as the 'Mouse game'.

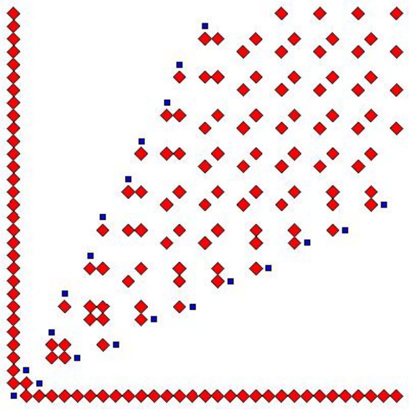
The (Mouse game)*



Moves and P-positions of (Mouse game)*



The Mouse trap := (the Mouse game)**



The Mouse's b-sequence is maximally b_1-superadditive

Remark

$\mathcal{P}$(Mouse game) is 'maximal' in the sense that the $b_1 - 1 = 1$ as in 2-superadditivity is attained everywhere, that is, for all $m, n > 0$ $b_{m+n} = b_m + b_n + 1$.

Is b_1-SAC necessary?

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Let G denote the invariant game with
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Let G denote the invariant game with $\mathcal{M}(G) = \{\{\lfloor \alpha n + \gamma \rfloor, \lfloor \beta n + \delta \rfloor\} \mid n \in \mathbb{N}\}$, where $\alpha = \sqrt{2}$, $\gamma = \sqrt{16.1} - 3\sqrt{2}$, $\beta = \sqrt{2} + 2$ and $\frac{\gamma}{\alpha} + \frac{\delta}{\beta} = 0$.

Is b_1-SAC necessary?

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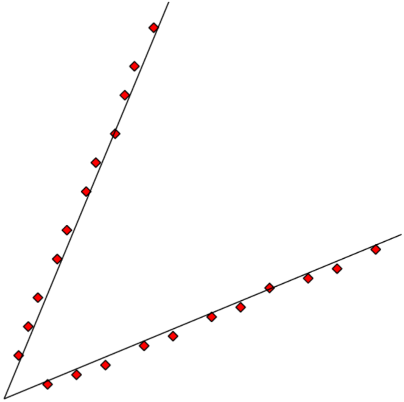
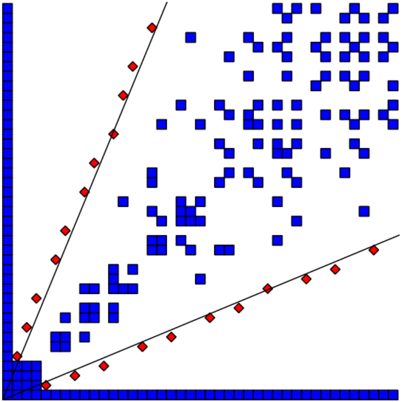
Let G denote the invariant game with $\mathcal{M}(G) = \{\{\lfloor \alpha n + \gamma \rfloor, \lfloor \beta n + \delta \rfloor\} \mid n \in \mathbb{N}\}$, where $\alpha = \sqrt{2}$, $\gamma = \sqrt{16.1} - 3\sqrt{2}$, $\beta = \sqrt{2} + 2$ and $\frac{\gamma}{\alpha} + \frac{\delta}{\beta} = 0$. In analogy with the P-positions of the Mouse game, a and b are complementary inhomogeneous Beatty sequences, but here $b_1 = 4$, $b_2 = 7$ gives $b_1 + b_1 > b_2$. Hence b is not super-additive.

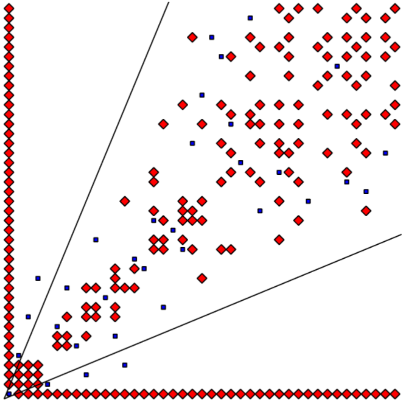
A non-super-additive b-sequence, $\mathcal{M}(G)$



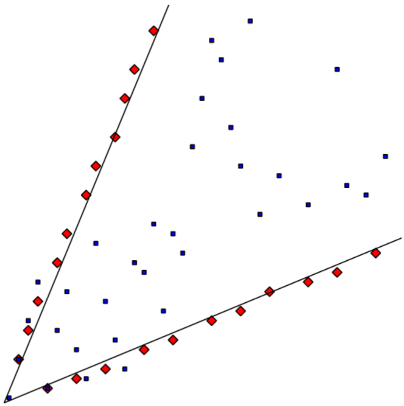
A non-super-additive b-sequence, $\mathcal{M}(G)$, $\mathcal{P}(G)$



A non-super-additive b-sequence, $\mathcal{M}(G^\star)$, $\mathcal{P}(G^\star)$



A non-super-additive b-sequence, $\mathcal{P}(G^{\star}) \neq \mathcal{M}(G)$.

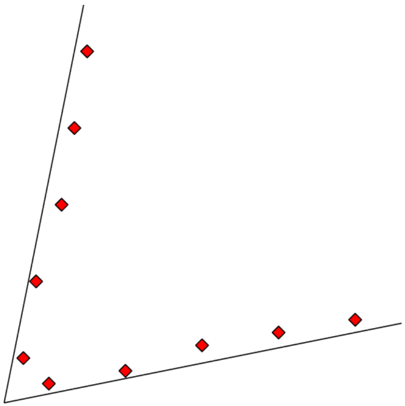


Is b_1-SAC necessary?

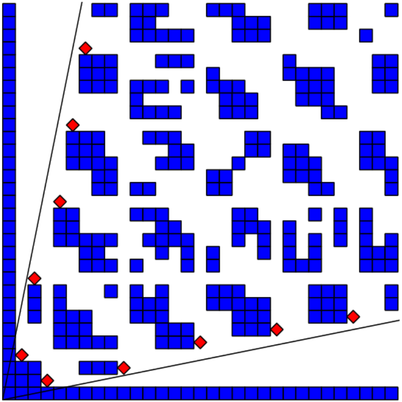
">" b_1-SAC

This example concerns the invariant game with moves of the form $(\lfloor \frac{6n+2}{5} \rfloor, 6n-3), (6n-3, \lfloor \frac{6n+2}{5} \rfloor)$. The set of ordered pairs $\{(1,3),(2,9),(4,15),(5,21),\dots\}$ is not b_1-super-additive. Namely, $b_1 + b_1 + b_2 = 15 = b_3$.

A non-b_1-super-additive b-sequence, $\mathcal{M}(G)$



A non-b_1-super-additive b-sequence



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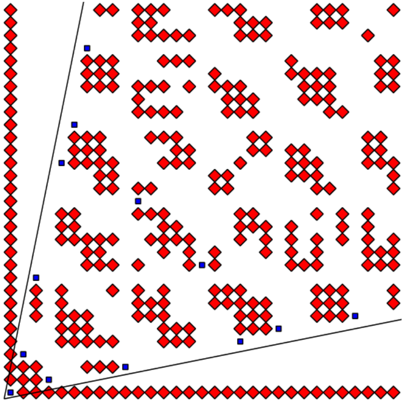

Other complementary pairs of sequences

Very special cases of increasing sequences

Both above examples were games on Complementary Inhomogeneous Betty Sequences, CIBS. (The latter had rational moduli, likewise Mouse game *.) Hence b were increasing.

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A confession

We do not know if there exists an invariant game with set of non-zero P-positions represented by CIBS (or even just infinite increasing sequences) but non-b_1-SAC.

Other complementary pairs of sequences

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Other complementary pairs of sequences

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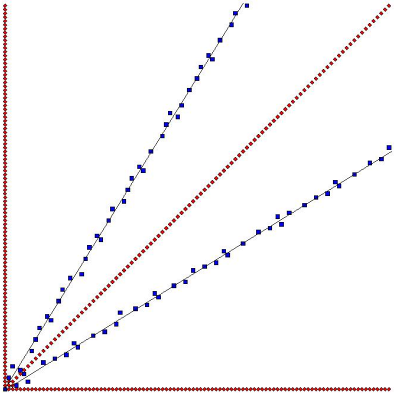
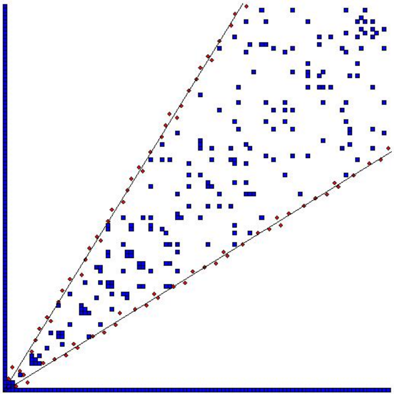
- Maharaja Nim: moves as in Wythoff Nim, but also $(1,2)$ and $(2,1)$,
- $(1,2)$GDWN: moves as in Maharaja Nim, but also $(t, 2t)$ or $(2t, t)$, $t \in \mathbb{N}$.

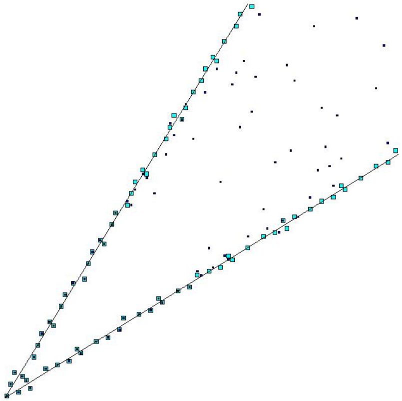
Maharaja Nim



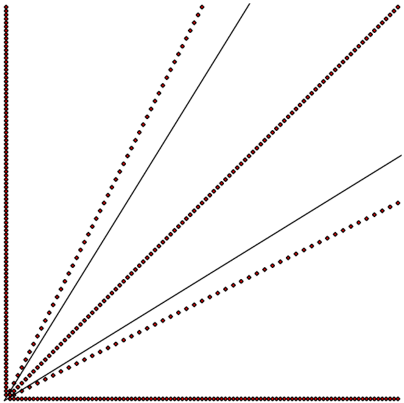
Maharaja Nim⋆: the b-sequence does not increase...



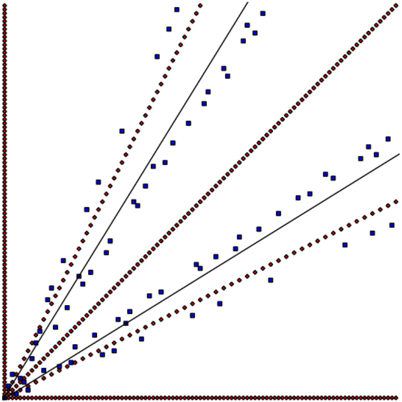
$\mathcal{P}$(Maharaja) and $\mathcal{P}$(Maharaja**) are disjoint



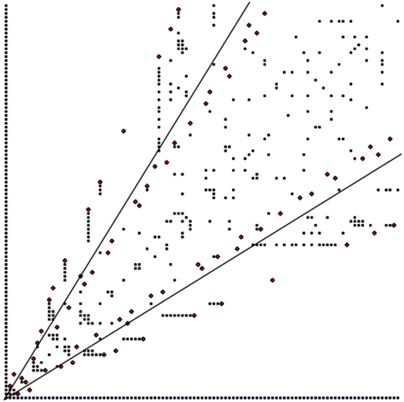
The moves of $(1,2)$GDWN and the lines ϕx and $\frac{x}{\phi}$



(1, 2)GDWN: The P-positions seem to “split”



Moves and P-positions of $(1, 2)$GDWN*



Questions about convergence of games

- Does $\mathcal{P}((1,2)\mathrm{GDWN}^{2k\star})$, $k \in \mathbb{N}_0$, converge?

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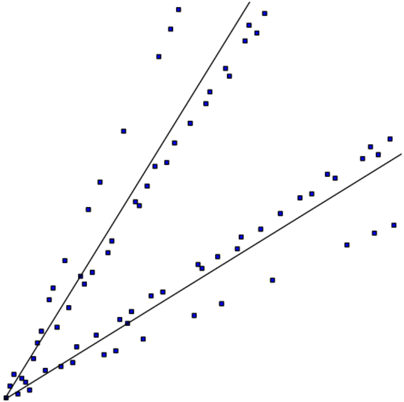
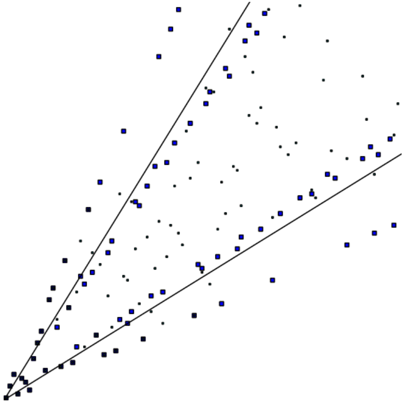
- Is there a k such that for all $k \leq l \in \mathbb{N}$, $\mathcal{P}((1,2)\text{GDWN}^{2l\star}) = \mathcal{P}((1,2)\text{GDWN}^{2k\star})$?

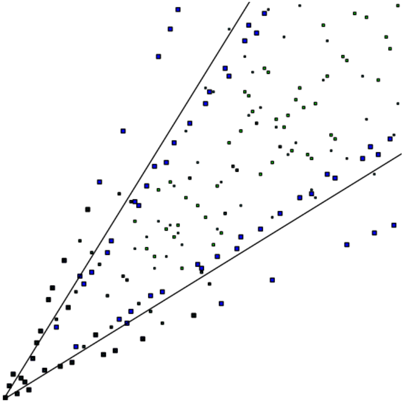
$\mathcal{P}((1,2)\text{GDWN})$



$\mathcal{P}((1,2)\text{GDWN}^{2\star})$



$\mathcal{P}((1,2)\mathrm{GDWN}^{4\star})$



Nim on several piles, Nim^k

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Is there a nice generalization of the $\star$-operator for

Partizan Subtraction games / Hackenbush?

Hackenbush*?

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- Is there a non-trivial partizan game G such that $G^{\star\star} = G$?