Impartial games on random graphs

Urban Larsson,
joint work with Johan Wästlund

Chalmers and University of Gothenburg, Sweden

October 11 2009

Table of contents

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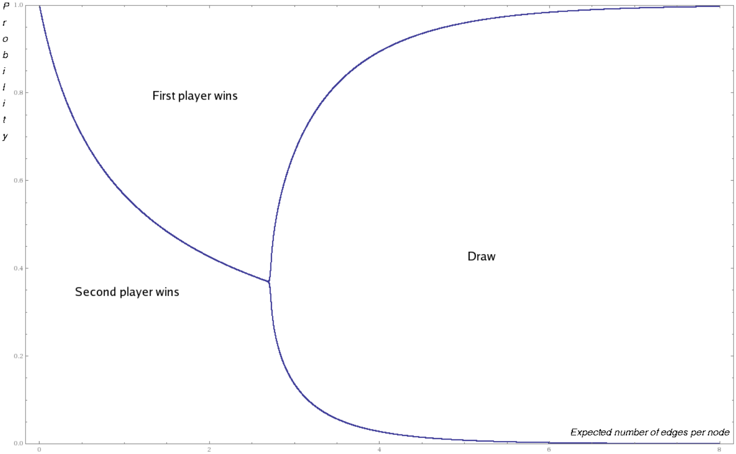

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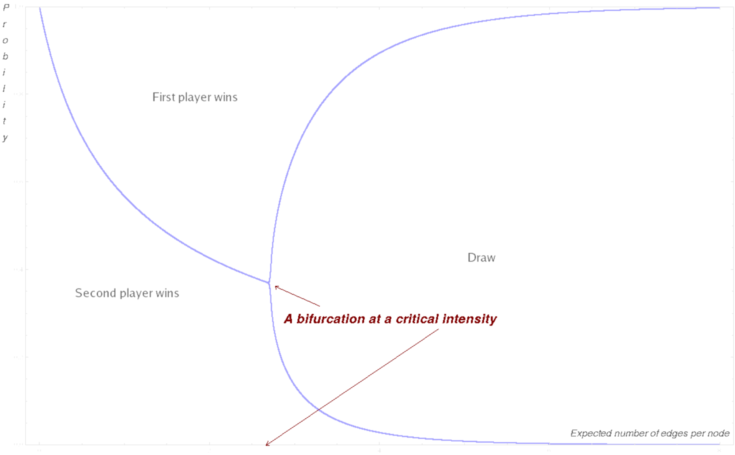

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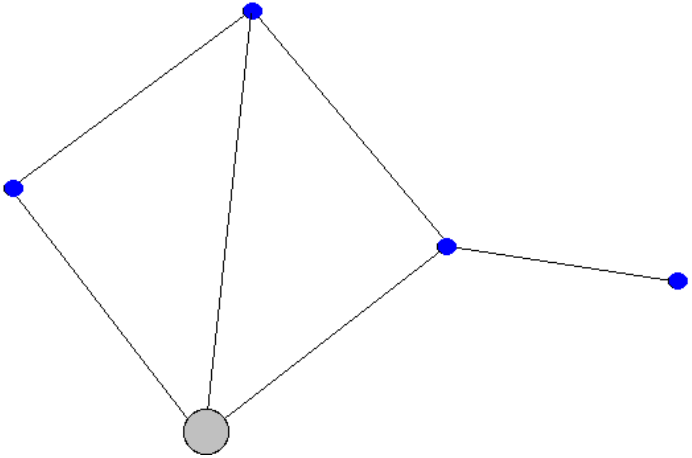

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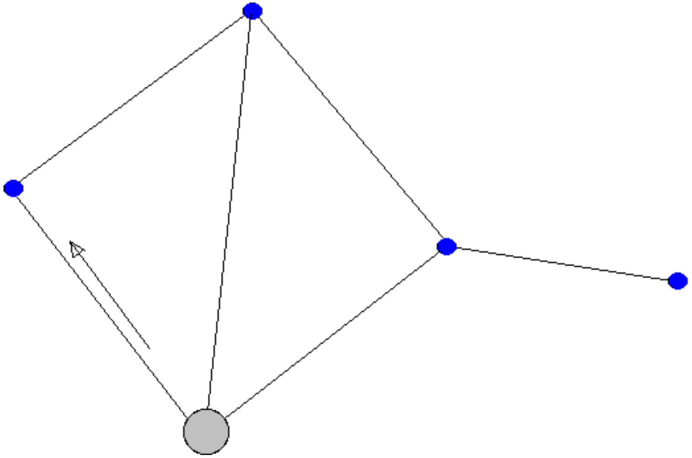

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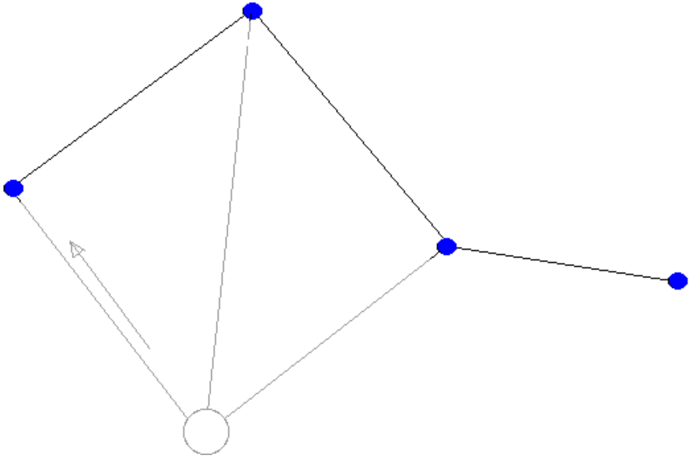

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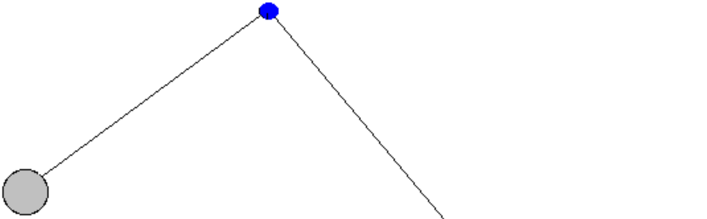

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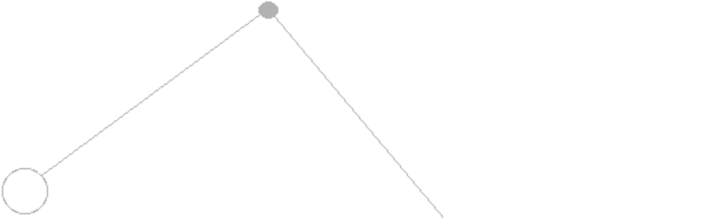

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- A.S. Fraenkel, E.R. Scheinerman and D. Ullman, (1993).

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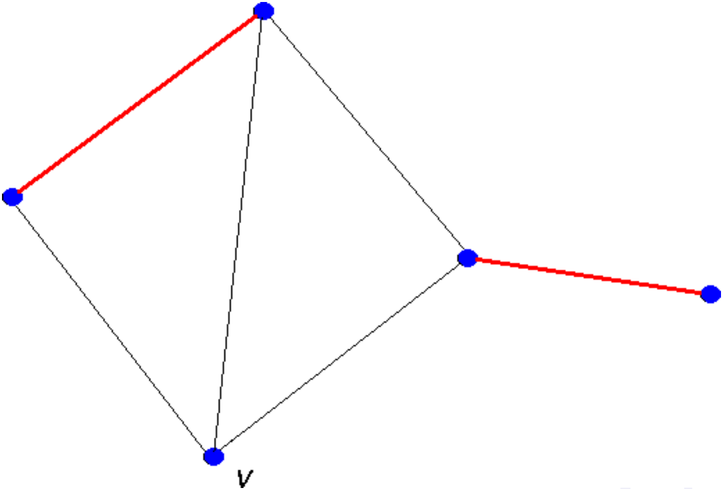

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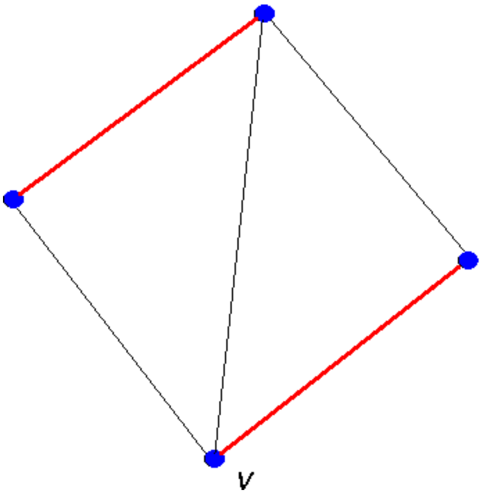

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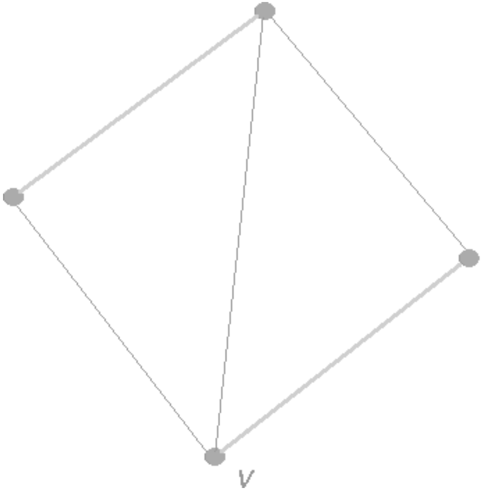

The Erdös-Rényi (ER) model

Let $n \in \mathbf{N}$ and $p \in [0, 1]$. Let $G(n, p)$ denote an ER-random graph on n nodes where an edge $\{x, y\}$ is present with probability p.

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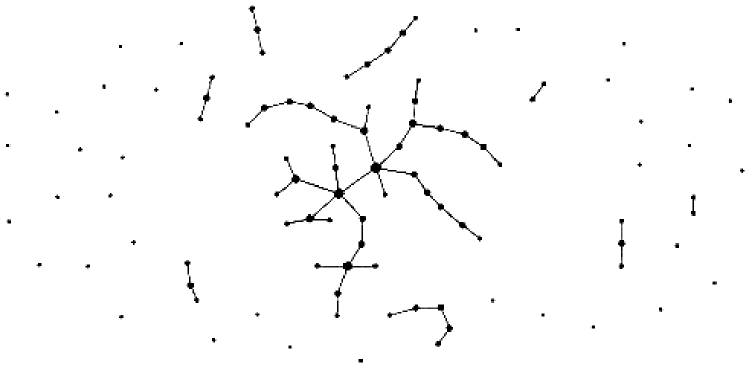
If $\lambda > 1$, almost surely $G(n,p)$ contains one giant component of size $\Theta(n)$. If $\lambda < 1$, the size of the largest connected component is $\Theta(log(n))$. The number of small cycles in a small component is small. Thus, locally, the graph resembles a Branching process.

An instance of G(100,0.01)



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- The probability of extinction is $f^{\infty}(a_0)$.

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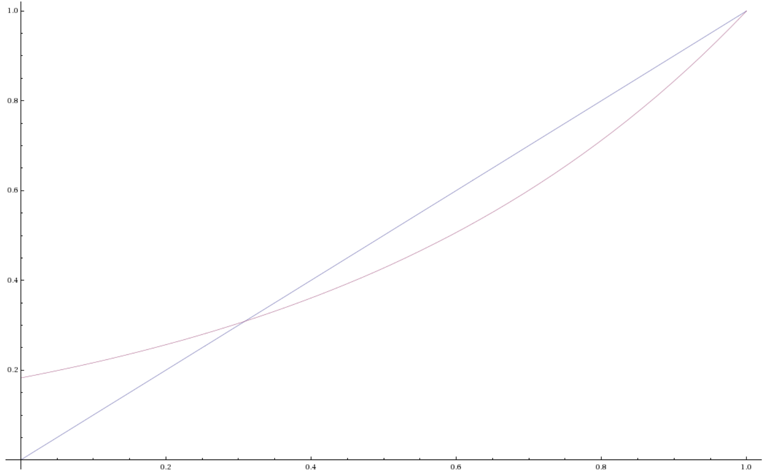
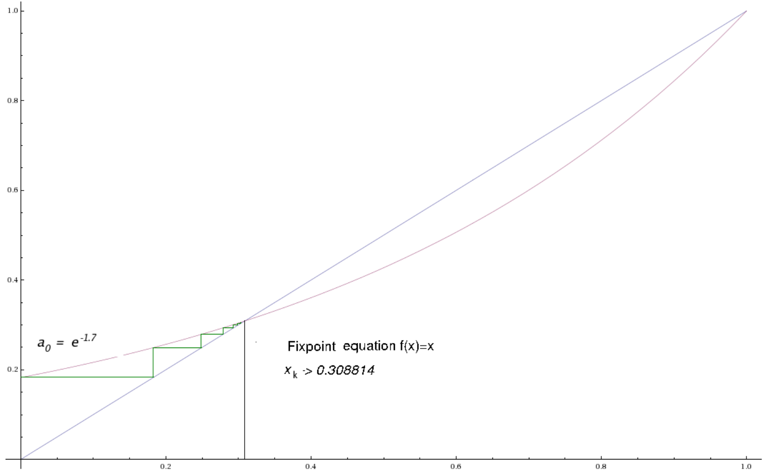
- This probability is given by the least positive solution to the fixpoint equation: $f(\alpha) = \alpha$.

$f_{1.7}(x)$



Iterating $x_{k+1} = f_{1.7}(x_k)$



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- Then $\lim p_k \to p$ and $\lim q_k \to q$.

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- cannot lose before game has started $q_0 = 1$.
- wins if there is no offspring: $p_1 = a_0 = e^{-\lambda}$.
- cannot lose in the first generation since it is the first players turn: $q_1 = q_0 = 1$.

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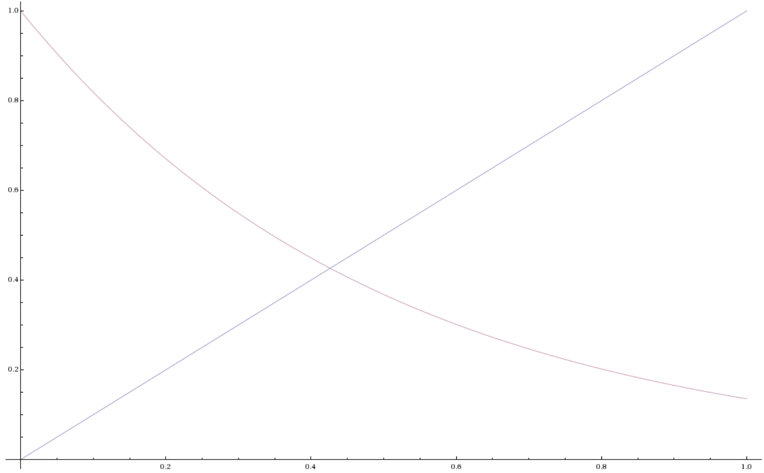
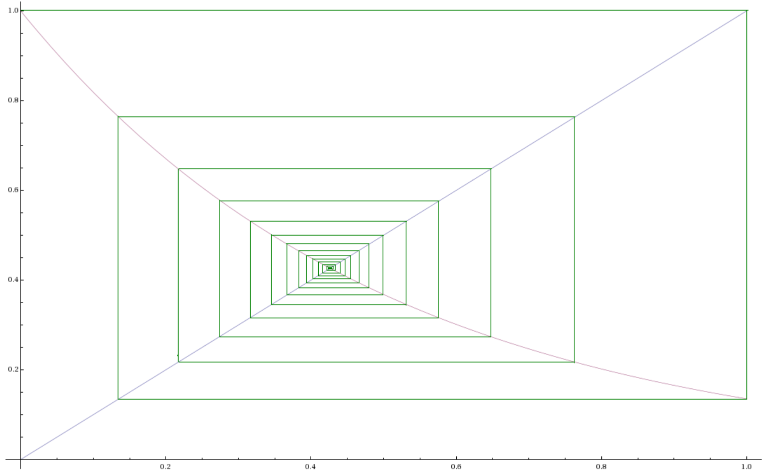
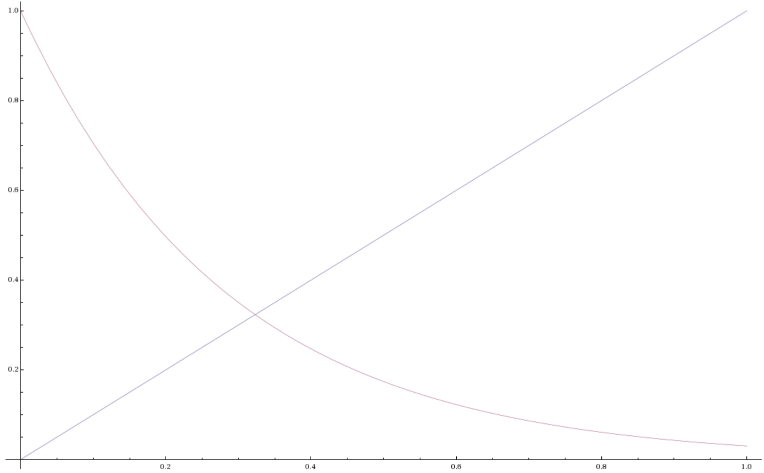


Iterating $p_k = e^{-2q_k}$ and $q_k = e^{-2p_k}$



x and $e^{-3.5x}$



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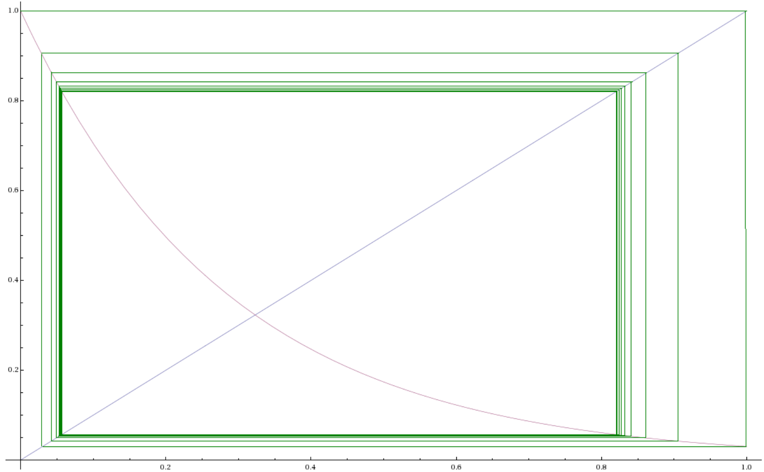

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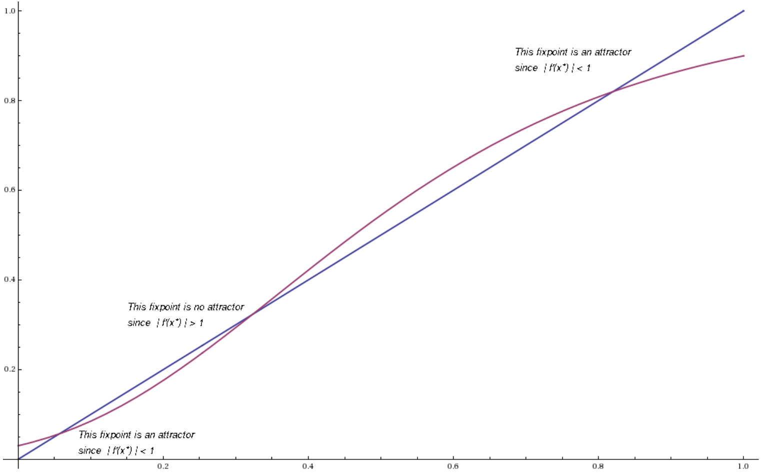
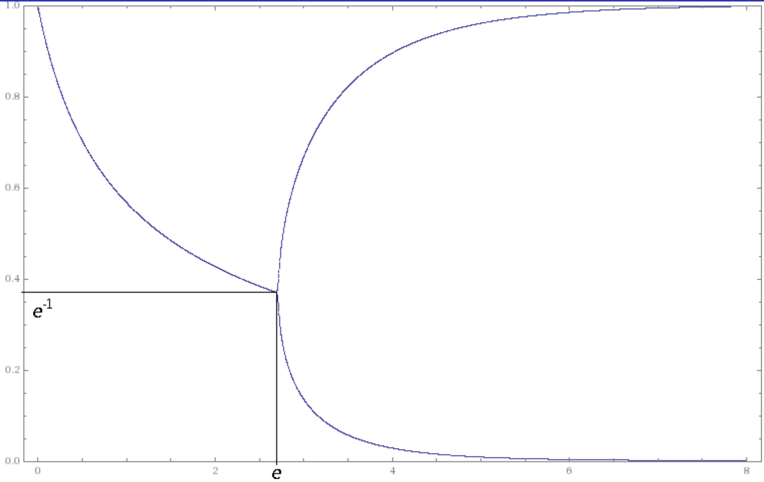
- But it is an attractor of period 2:

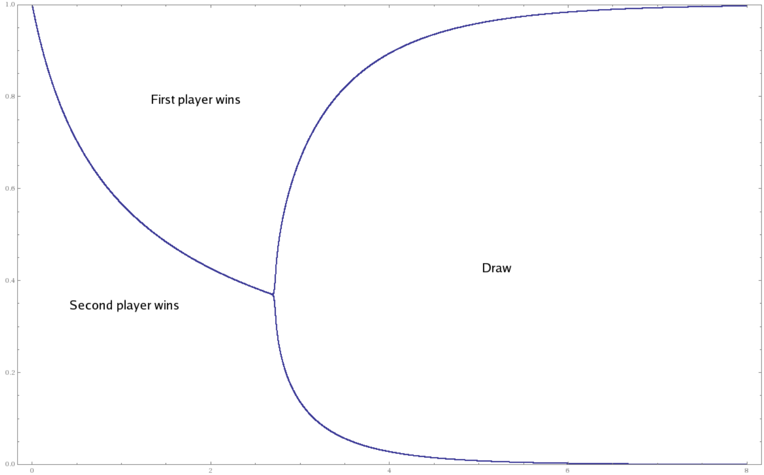
x and $e^{-3.5e^{-3.5x}}$



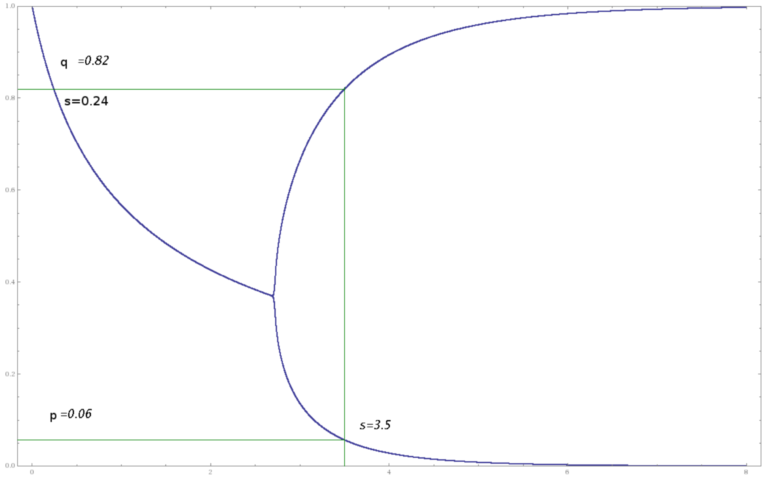
A bifurcation at $\lambda = e$



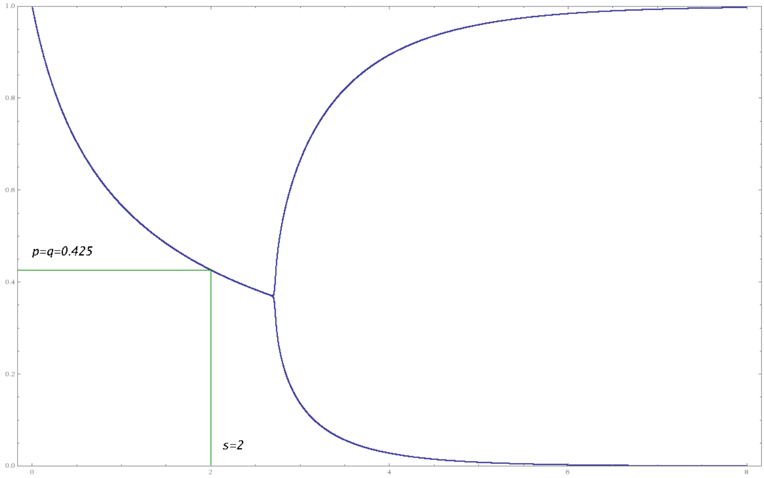
A game theoretical interpretation



$p < q$ if $\lambda > e$



$p = q$ if $\lambda \leq e$



Why a bifurcation at $\lambda = e$?

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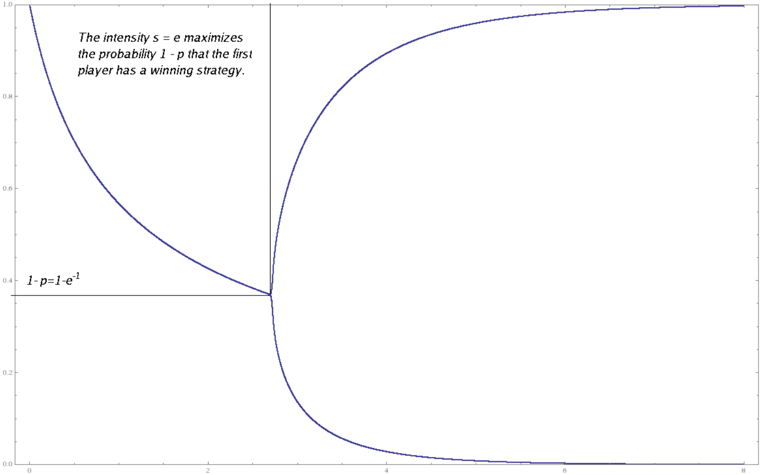
- But $g'(\alpha) = -\lambda e^{-\lambda\alpha} = -\lambda\alpha \geq -1$;
- This gives $\lambda \leq e$. At the critical intensity the probability for a second player win is $\alpha = \frac{1}{e}$. □

When does the first player win?



The expected size of a maximum matching in $G(n, p)$

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$G(n, p)$ and a pseudo-draw

Suppose we play a game of UVG on a finite graph with n nodes. Then, if no player can force a win within $\sqrt{(}\log(n))$ moves, we define the outcome of the game as a pseudo-draw.

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A blocking maneuver

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Let $k \in \mathbf{N}$. The rules of k-blocking UVG are as UVG with the following twist: Before the next player moves, the previous player may block off at most $k-1$ edges and declare them as non-slidable.

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So 1-blocking UVG = UVG.

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Hence, for 2-blocking UVG, if a_i is Poissonian, we get:

$$q = (1 + \lambda p)e^{-\lambda p}$$

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$$p = (1 + \lambda q)e^{-\lambda q}$$

and so for this game the critical intensity $\lambda_0 = \frac{e^{\phi}}{\phi}$, where $\phi = \frac{1+\sqrt{5}}{2}$.

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and so for this game the critical intensity $\lambda_0 = \frac{e^{\phi}}{\phi}$, where $\phi = \frac{1+\sqrt{5}}{2}$. At this intensity and below, the probability for a draw is 0. The probability for a player B win at this intensity is $\frac{\phi^2}{e^{\phi}}$.

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- The critical intensity for k-blocking GWUVG is $\lambda_0 = \frac{k!e_0^x}{x_0^k}$.

  The probability for a Second player win is $\alpha = \frac{x_0^{k+1}}{k!e_0^x}$.

Other distributions?

Let a_i be uniformly distributed on $0, 1, \ldots, N-1$ so that $a_i = 1/N$ if $i \in \{0, 1, \ldots, N-1\}$, and zero otherwise. Denote UVG on this GW process N-GW.

Theorem

The probability for a draw on N-GW with uniform distribution is zero for all $N \geq 0$. For $N = 2, 3$ the second player wins with probability $2/3$ and $3 - \sqrt{6}$ 0.55. For $N > 3$ the probability for a first player win is > 0.5

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Is this the end of the story of random 'bifurcation games'?

Wighted Heads($=0$ children) and tails ($=2$)?