2-pile Nim with a Restricted Number of Move-size Imitations

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Seminariet i diskret matematik, GU

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To my daughter Hanna, whom I have played hundreds of 'imitation games' with.

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At last, and in a slightly different setting, we are going to look at when there is a useful second player strategy for a game. When can he be "certain to learn" the winning strategy while playing (given that the first player knows the strategy).

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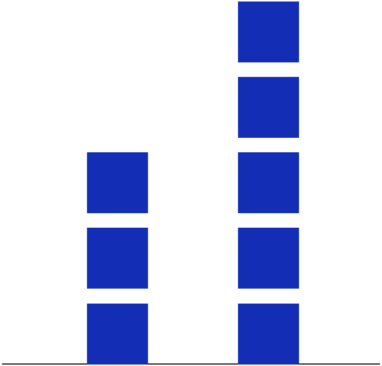

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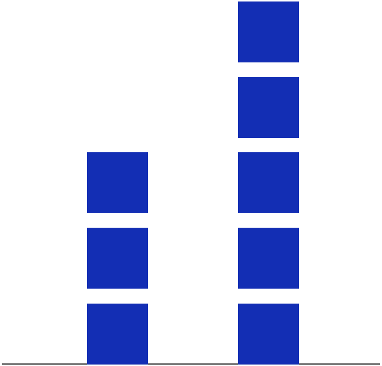

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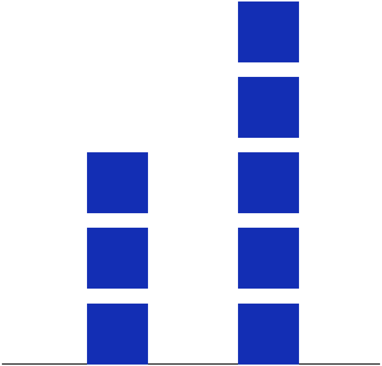

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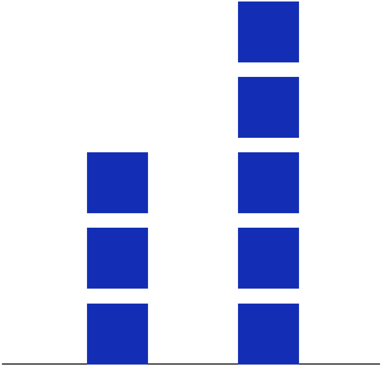

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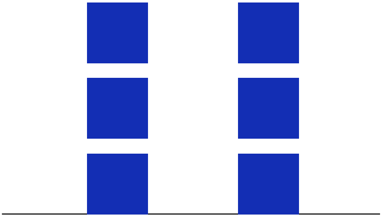

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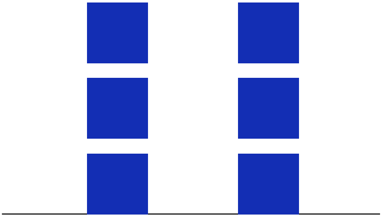

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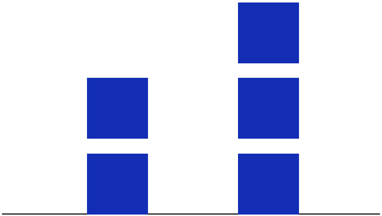

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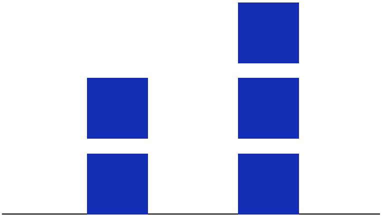

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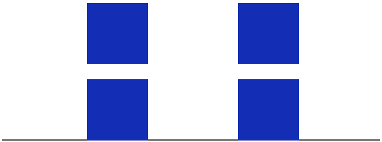

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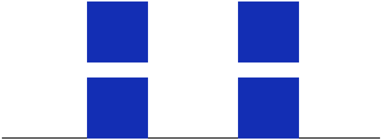

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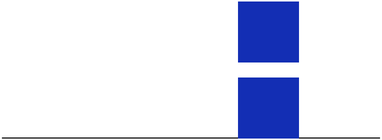

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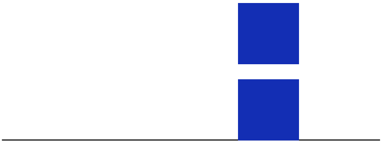

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A position from which the player who made the last move, the previous player, can win given best play, is called a P-position. A position from which the next player can win, given best play, is called an N-position. Given a game G, denote with $\mathcal{P}_G$, the set of all P-positions of G.

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- a position is an N-position iff there is a P-position in its set of followers.

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Our games can be played on two piles of tokens and where the players may only remove tokens from the top of the piles (given some rules). Such games are known as (2-pile) take away games or Nim-like games.

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Remark: The leading pile might change from one move to the next.

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But first some notation:

The default color of a token is blue. A token is green if removal of the token (including the ones above) implies that an *imitation counter* is increased by one. A token is yellow if it, for the reason of the previous player's move, may not be removed. The imitation counter is drawn as a black square.

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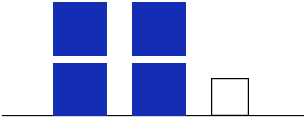

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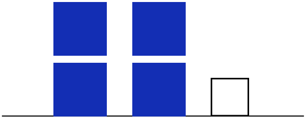

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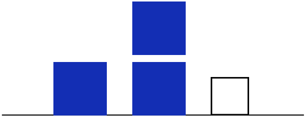

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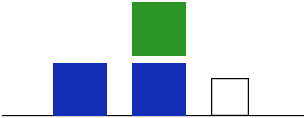

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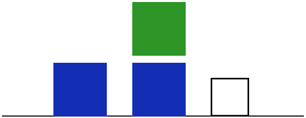

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- This time, the second player may not imitate, in fact he can't move at all.
- So (2,2) is a next player winning position...unlike 2-pile Nim.

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The game is maybe more known as the impartial game "Corner the queen" (Rufus P. Isaacs, 1960), where the two players alternate to move one single queen - aiming to get to the bottom left corner of a (large) chessboard. The distance to this corner must by each move decrease and precisely the ordinary chess-queen moves are allowed. The winner is the player who first puts the queen in the corner.

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A variation of Wythoff's game

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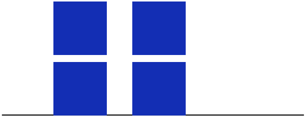
Example: One diagonal option may be blocked



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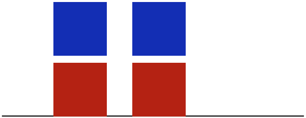

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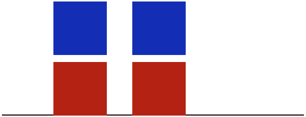

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- The first player moves to (1,1) and makes the obvious block;
- The second player may remove either one of the tokens, but not both;
- and since a player may not block off a single token,
- the first player wins, so (2,2) is a next player winning position...
- ...just like for (2,1)-Imitation Nim...

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The Wythoff-pairs

The P-positions of the game of Wythoff Nim, denoted by $\mathcal{P}_W$, are intuitively given by the rule "the least number - the least difference":

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$(0,0), \{1,2\}, \{3,5\}, \{4,7\}$, etc

Partitions of the natural numbers

Beatty's theorem (1927) says:

Theorem

If x and y are positive irrational numbers such that $\frac{1}{x} + \frac{1}{y} = 1$ *then* $\{\lfloor ix \rfloor \mid i \in \mathbf{N}\} \sqcup \{\lfloor iy \rfloor \mid i \in \mathbf{N}\} = \mathbf{N}$.

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As a consequence of Beattys theorem, the sequence of "Wythoff-pairs" that we denote with

$$\mathcal{P}_W = (\{a_i, b_i\})_{i=0}^{i=\infty},$$

$0 \leq a_i \leq b_i$, can be generated in polynomial time. It exhibits some beautiful properties:

Number theoretical properties of the Wythoff-pairs

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(v) *viewed as a permutation of the natural numbers,* $\mathcal{P}_W$ *is the unique permutation satisfying the properties (i), (ii) and (iii), (Knape, Larsson 2004).*

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III. but before the next player makes his move, the previous player may declare at most $k - 1$ of the type-II moves, with $i = j$, as blocked options (Hegarty, Larsson).

The set $\mathcal{P}_{W_{k,m}}$

One can generate the P-positions of (k, m)-Wythoff Nim via a so called minimal exlusive algorithm. We use the following standard notation: For X a set of natural numbers, $\operatorname{mex} X$ denotes the least natural number $\notin X$.

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Given positive integers k and m, the integer sequences (a_i) and (b_i) are defined as follows: $a_0 = b_0 = 0$ and for $i > 0$

$$\begin{aligned} a_i &:= \operatorname{mex}\{a_j, b_j \mid 0 \leq j < i\}; \\ b_i &:= a_i + \delta_i, \end{aligned}$$

where $\delta_i = \delta_i(k, m) := \lfloor \frac{i}{k} \rfloor m$.

Arithmetic properties of $W_{k,m}$

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(vi) *here, (a_i) and (b_i) can be described completely by Beatty sequences only for special cases, namely if $k \mid m$.*

Recent progress

In the Appendix of my paper '2-Pile Nim with a restricted number of move-size imitations', P. Hegarty shows that if $k > 0$ and $m = 1$, the sequences are "close to" Beatty sequences. We had previously conjectured that this holds for all $k > 0$...

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Duchêne's game

Let r be a positive integer and let D_r be the game defined as follows: Move as in Wythoff Nim, but any diagonal move, say $(a, b) \to (c, d)$, has to be of the form $d - b \equiv c - a \pmod{r}$.

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Proposition

Let D_2 be the game defined as above. Put $\mathcal{P}_{D_2} = (\{p_n, q_n\})$, where for all $n \geq 0, p_n \leq q_n$ and let $\mathcal{P}_{W_{4,2}} = (\{a_n, b_n\})$ where $a_n \leq b_n$. Then, provided the sequences are written in increasing order, $(a_n) = (p_n), (b_{4n}) = (q_{4n}), (b_{4n+1}) = (q_{4n+1}), (b_{4n+2}) = (q_{4n+3})$ and $(b_{4n+3}) = (q_{4n+2})$.

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'Geometrical extensions of Wythoff's game', E. Duchêne and S. Gravier, 2007. Private communication with Eric, 12 April: No generalization to the Proposition is known.

A dynamic counting of P-positions of $W_{k,m}$

Definition

Let (a, b) be a P-position as in $W_{k,m}$. Then

$$\xi((a,b)) := \#\{(i,j) \mid i \geq a, j - i = b - a, (i,j) \in \mathcal{P}_{W_{k,m}}\}.$$

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Notice that for a winning strategy at least $k - \xi((a, b))$ positions must be blocked off.

Imitations and how to count them

Suppose the previous player removed x tokens from a leading pile. Then if the next player removes $x + i$ tokens from the other pile, where $0 \leq i < m$, he *m-imitates* (or just imitates) the previous player's move.

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Suppose the current position of a 2-pile Nim-like game is X and the last two moves are $Z \to Y \to X$. Then put

$$L(X) := L(Z) + 1$$

if $Y \to X$ imitates $Z \to Y$. Otherwise $L(X) := 0$.

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Note: An option $X \to Y$ may be viewed as an imitation although no move has yet been made.

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Note: At the Integers 2007 conference, A. Fraenkel suggested the name 'Limitation Nim' whenever we discuss the case $k = 1$.

The game of (k, m)-Imitation Nim

Definition

Let k and m be two positive integers. The game of (k, m)-Imitation Nim (or just Imitation Nim) is a take-away game on two piles of tokens, where the players

- move as in 2-pile Nim, but
- with the restriction that no more than $k-1$ m-imitations in a strict sequence by one and the same player are permitted;
- More precisely, a player may not move to an option, say X, if it would imply $L(X) = k$.

Note: At the Integers 2007 conference, A. Fraenkel suggested the name 'Limitation Nim' whenever we discuss the case $k = 1$. Touché.

Comparing the number of options for modified games

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Notation:

Let G be a game. Let us introduce the following "non-standard" terminology. A position is dynamic if it is impossible to tell whether it is a P- or N-position without any information on the previous move(s). If a position is not dynamic it is non-dynamic.

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- dynamic positions;
- non-dynamic
  - P-positions;
  - N-positions.

Note: This above notation is only relevant for move-size dynamic games. Each position of Wythoff Nim is clearly non-dynamic. The winning nature of a dynamic position may be classified for a specific 'partie' but not for a game position in general, where the value of the counter L is 'indecidable'.

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Maybe, a 'partie'-theory has yet to be developed...

Stable and free positions of blocking Wythoff Nim

Suppose that $X = (a, b)$ is a P-position of (k, m)-Wythoff Nim. If in addition the following statement holds,

if $(c, d) \in \mathcal{P}_{W_{k,m}}$ with $b - a = d - c$ then $c \geq a$;

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- *if X is P-free then X is a P-position of Imitation Nim iff $L(X) < \xi(X)$;*
- *if X is N-free then X is a P-position of Imitation Nim iff there is a P-position of (k,m)-Wythoff Nim, say Y, such that $L(Y) \geq \xi(Y)$ and $X \rightarrow Y$ is an m-imitation.*

For $k = m = 1$, we may look at an illustration of the "N-stable case", given inductively the "P-stable case". ▸ Example

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(iii) Imitation Nim - yes, by "carefully" imitating the previous player's moves and being aware that each $W_{k,m}$ P-position has difference of pileheights $\equiv 0 \pmod{m}$.

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- if he can/may not imitate, then it suffices to move to a position (a, c) where c is the largest permissable number of tokens of the non-leading pile. Indeed this move 'forces' the first player to move $(a, c) \to (a', c)$ where a' is the greatest integer less than a such that there is a P-position (a', b') of (k, m)-Wythoff Nim (and P-free iff $k > 1$). Then for this particular game we get $L((a', b')) = 1$.

Example: (2,3)-Imitation Nim

Now, put $k = 2$ and $m = 3$. The first few P-positions of $(2, 3)$-Wythoff Nim are:

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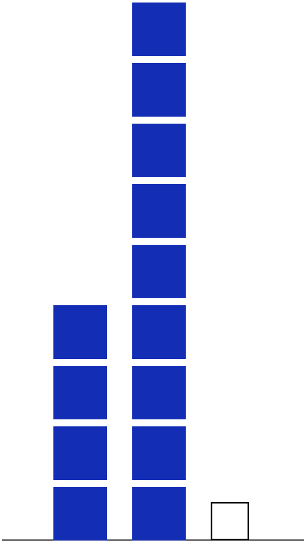
These positions are well-known to the first player, but not to the second player. Suppose the first player is about to move from $(4, 9)$, an N-stable position.

$\mathcal{P}_{W_{2,3}} = \{(0,0),(1,1),(2,5),(3,6),(4,10),(7,13),\ldots\}$



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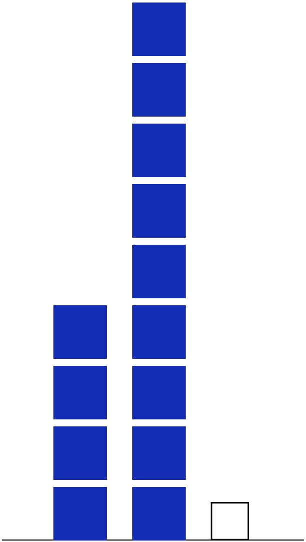

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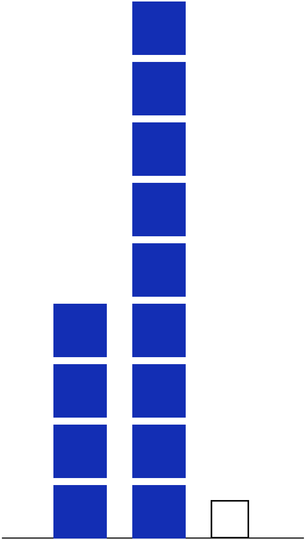

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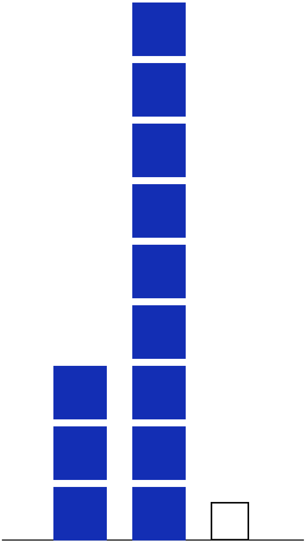

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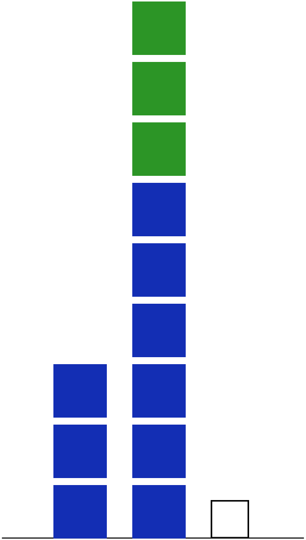

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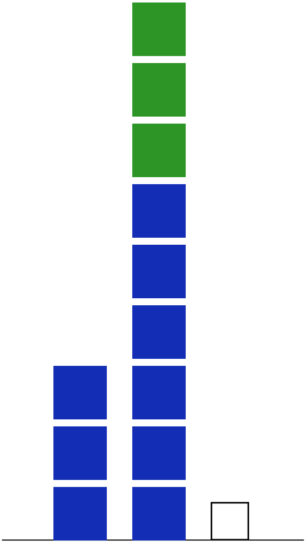

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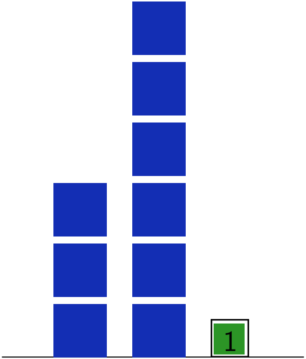

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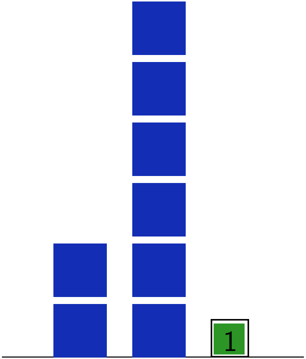

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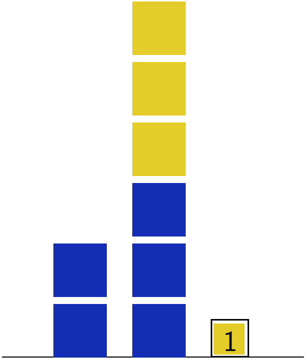

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- The only winning move from (3,6);
- The second player may not imitate a second time; In particular he can not reach $(2, 5)$ (but $5 - 2 \equiv 0 \pmod 3$);

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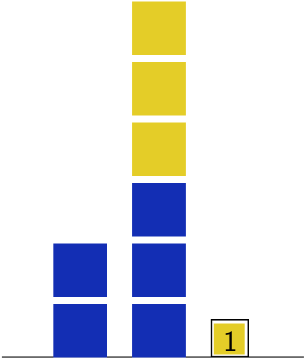

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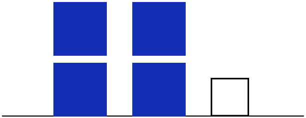

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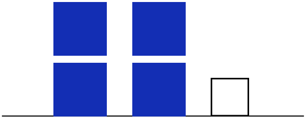

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- Limitation Nim = (1,1)-Imitation Nim is the game where the next player may not imitate the previous player's most recent move. This game has the same P-positions as Wythoff Nim, if one only regards the starting positions.

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- A "wider m-diagonal" of Wythoff Nim corresponds to an "m-relaxed" notion of an imitation. What is more, there is a precise dynamic correspondence between the winning positions of the games (k,m)-Wythoff Nim and (k,m)-Imitation Nim. This relationship constitutes our main theorem.

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- "Imitation provides means for learning". Our Imitation games offer nice second player strategies for learning the first player's winning strategy. And indeed the main ingreadient for such a strategy is to imitate the first players moves. The setting of our blocking games does not allow a corresponding second player strategy, since the first player may 'conceal' P-positions.

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Thank you!

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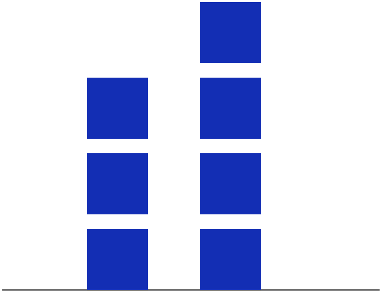

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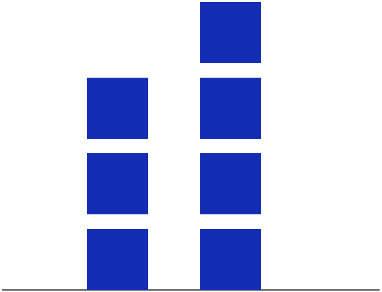

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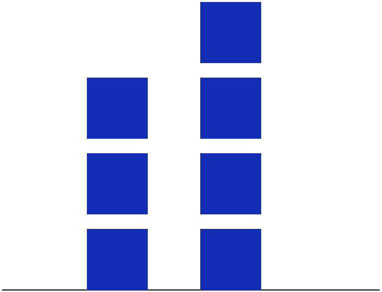

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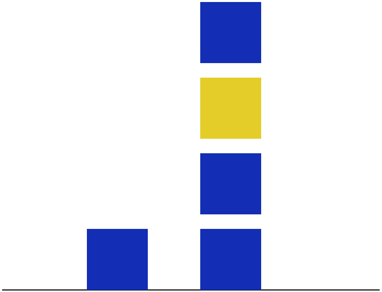

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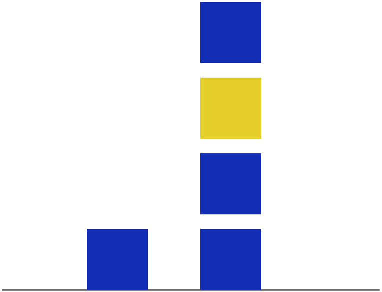

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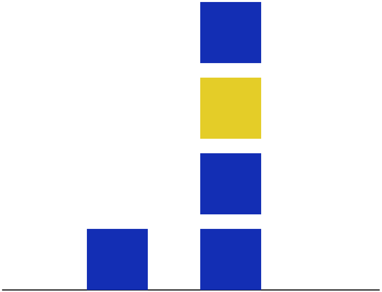

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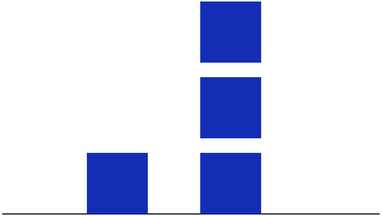

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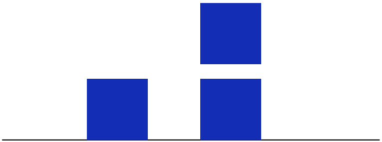
- The first player responds with a move to the P-stable position (1,2)... wins.



From this argument it is not hard to see that all N-stable positions of form (I) belongs to $\mathcal{N}_{I_{1,1}}$. With a little extra work the argument can be extended to general k and m. ▸ return

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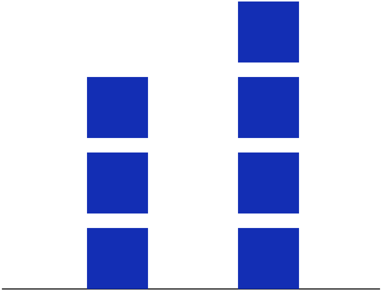

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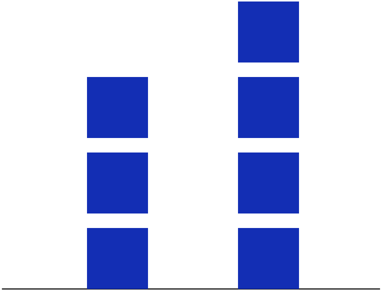

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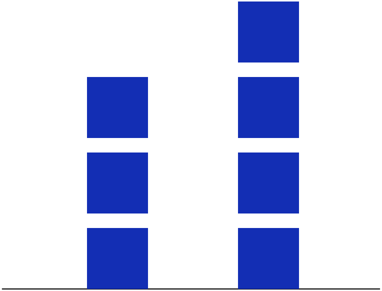

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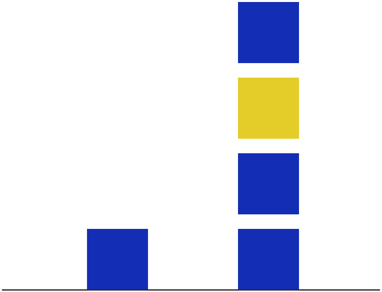

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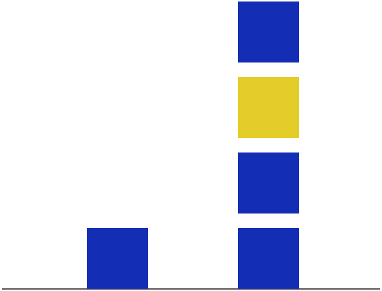

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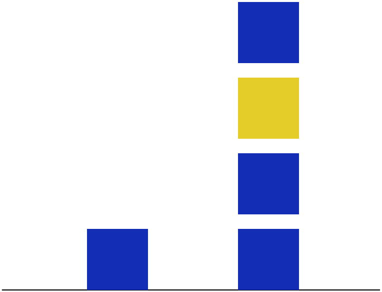

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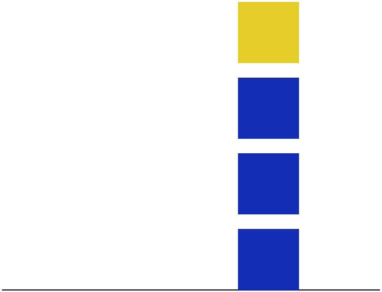

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